

FI-modules: a new approach to stability for S_n -representations

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Abstract

In this paper we introduce and develop the theory of FI-modules. We apply this theory to obtain new theorems about:

- the cohomology of the configuration space of n distinct ordered points on an arbitrary (connected, oriented) manifold;
- the diagonal coinvariant algebra on r sets of n variables;
- the cohomology and tautological ring of the moduli space of n -pointed curves;
- the space of polynomials on rank varieties of $n \times n$ matrices;
- the subalgebra of the cohomology of the genus n Torelli group generated by H^1 ;

and more. The symmetric group S_n acts on each of these vector spaces. In most cases almost nothing is known about the characters of these representations, or even their dimensions. We prove that in each fixed degree the character is given, for n large enough, by a polynomial in the cycle-counting functions that is independent of n . In particular, the dimension is eventually a polynomial in n .

FI-modules are a refinement of Church–Farb’s theory of representation stability for representations of S_n . In this framework, a complicated sequence of S_n -representations becomes a single FI-module, and representation stability becomes finite generation. FI-modules also shed light on classical results. From this point of view, Murnaghan’s theorem on the stability of Kronecker coefficients is not merely an assertion about a list of numbers, but becomes a structural statement about a single mathematical object.

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1 Introduction

In this paper we develop a framework by which we can deduce strong constraints on naturally occurring sequences of S_n -representations using only elementary structural symmetries. These structural properties are encoded by objects we call *FI-modules*. Let FI be the category whose objects are finite sets and whose morphisms are injections. This is equivalent to the category whose objects are natural numbers \mathbf{n} , with the morphisms $\mathbf{m} \rightarrow \mathbf{n}$ being the injections from $\{1, \dots, m\}$ to $\{1, \dots, n\}$.

Definition 1.1 (FI-module). An *FI-module* over a commutative ring k is a functor V from FI to the category of modules over k . We denote the k -module $V(\mathbf{n})$ by V_n .

Since $\text{End}_{\text{FI}}(\mathbf{n}) = S_n$, any FI-module V determines a sequence of S_n -representations V_n with linear maps between them respecting the group actions. One theme of this paper is the conceptual power of encoding this large amount of (potentially complicated) data into a single object V .

Many of the familiar notions from the theory of modules, such as submodule and quotient module, carry over to FI-modules. In particular, there is a natural notion of *finite generation* for FI-modules.

Definition 1.2 (Finite generation). An FI-module V is *finitely generated* if there is a finite set S of elements in $\coprod_i V_i$ so that no proper sub-FI-module of V contains S ; see Definition 2.15.

It is straightforward to show that finite generation is preserved by quotients, extensions, tensor products, etc. Moreover, finite generation also passes to submodules when k contains \mathbb{Q} (see §2.7), so this is quite a robust property. Our first main theorem is the following.

Theorem 1.3 (Polynomiality of dimension). *If V is a finitely generated FI-module over a field of characteristic 0, there is an integer-valued polynomial $P \in \mathbb{Q}[T]$ and some $N \geq 0$ so that*

$$\dim(V_n) = P(n) \quad \text{for all } n \geq N.$$

Examples of FI-modules. Theorem 1.3 would not be of much use if FI-modules were rare. Fortunately, FI-modules are ubiquitous. To illustrate this we present in Table 1 a variety of examples of FI-modules that arise in topology, algebra, combinatorics and algebraic geometry. In the course of this paper we will prove that each entry in this list is a finitely generated FI-module. The exact

<u>FI-module $V = \{V_n\}$</u>	<u>Description</u>
1. $H^i(\text{Conf}_n(M); \mathbb{Q})$	$\text{Conf}_n(M)$ = configuration space of n distinct ordered points on a connected, oriented manifold M (§4)
2. $R_J^{(r)}(n)$	$J = (j_1, \dots, j_r)$, $R^{(r)}(n) = \bigoplus_J R_J^{(r)}(n)$ = r -diagonal coinvariant algebra on r sets of n variables (§3.2)
3. $H^i(\mathcal{M}_{g,n}; \mathbb{Q})$	$\mathcal{M}_{g,n}$ = moduli space of n -pointed genus $g \geq 2$ curves (§5.1)
4. $\mathcal{R}^i(\mathcal{M}_{g,n})$	i^{th} graded piece of tautological ring of $\mathcal{M}_{g,n}$ (§5.1)
5. $\mathcal{O}(X_{P,r}(n))_i$	space of degree i polynomials on $X_{P,r}(n)$, the rank variety of $n \times n$ matrices of P -rank $\leq r$ (§3.4)
6. $G(A_n/\mathbb{Q})_i$	degree i part of the Bhargava–Satriano Galois closure of $A_n = \mathbb{Q}[x_1, \dots, x_n]/(x_1, \dots, x_n)^2$ (§3.3)
7. $H^i(\mathcal{I}_n; \mathbb{Q})_{\text{alb}}$	degree i part of the subalgebra of $H^*(\mathcal{I}_n; \mathbb{Q})$ generated by $H^1(\mathcal{I}_n; \mathbb{Q})$, where \mathcal{I}_n = genus n Torelli group (§5.2)
8. $H^i(\text{IA}_n; \mathbb{Q})_{\text{alb}}$	degree i part of the subalgebra of $H^*(\text{IA}_n; \mathbb{Q})$ generated by $H^1(\text{IA}_n; \mathbb{Q})$, where IA_n = Torelli subgroup of $\text{Aut}(F_n)$ (§5.2)
9. $\text{gr}(\Gamma_n)_i$	i^{th} graded piece of associated graded Lie algebra of many groups Γ_n , including \mathcal{I}_n , IA_n and pure braid group P_n (§5.3)

Table 1: Examples of finitely generated FI-modules

definitions of each of these objects will be given later in the paper. Any parameter here not equal to n should be considered fixed and nonnegative.

Except for a few special (e.g. $M = \mathbb{R}^d$) and low-complexity (i.e. small i , d , g , J , etc.) cases, the dimensions of the vector spaces (1)–(9) in Table 1 are not known, or even conjectured. Exact computations seem to be extremely difficult. By contrast, the following result gives an answer, albeit a non-explicit one, in all of these cases.

Corollary 1.4. *Let $\{V_n\}$ be any of the sequences of vector spaces (1)–(9) in Table 1. Then there exists an integer N and an integer-valued polynomial $P \in \mathbb{Q}[T]$ such that*

$$\dim(V_n) = P(n) \quad \text{for all } n \geq N.$$

Corollary 1.4 is a direct consequence of our general theory. It is easy to check that each sequence $\{V_n\}$ in Table 1 comes from an FI-module V , and the machinery we develop makes it straightforward to check that each V is finitely generated; the corollary then follows by applying Theorem 1.3. As a contrasting example, the dimension of $H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ grows exponentially with n , where $\overline{\mathcal{M}}_{g,n}$ is the Deligne–Mumford compactification of the moduli space of n -pointed genus g curves. Although the cohomology groups $H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ do form an FI-module, this FI-module is not finitely generated.

Apart from a few special cases, we do not know how to specify the polynomials produced by Corollary 1.4, although we can give explicit upper bounds for their degree. A primary obstacle is that

our theorems on finite generation depend on the Noetherian property of FI-modules proved in §2.7, and such properties cannot in general be made effective.

Character polynomials. All of the vector spaces in Table 1 have additional structure — each admits a natural action of the symmetric group S_n . As noted above, any FI-module V provides a linear action of $\text{End}_{\text{FI}}(\mathbf{n}) = S_n$ on V_n . To prove Theorem 1.3, we actually prove a stronger description of the V_n as S_n -representations, which we now describe.

For each $i \geq 1$ and any $n \geq 0$, let $X_i: S_n \rightarrow \mathbb{N}$ be the class function defined by

$$X_i(\sigma) = \text{number of } i\text{-cycles in the cycle decomposition of } \sigma.$$

Polynomials in the variables X_i are called *character polynomials*. Though perhaps not widely known, the study of character polynomials goes back to work of Frobenius, Murnaghan, Specht, and Macdonald (see e.g. [Mac, Example I.7.14]).

It is easy to see that for any $n \geq 1$ the vector space of class functions on S_n is spanned by character polynomials, so the character of any representation can be described by such a polynomial. For example, if $V \simeq \mathbb{Q}^n$ is the standard permutation representation of S_n , the character $\chi_V(\sigma)$ is the number of fixed points of σ , so $\chi_V = X_1$. If $W = \bigwedge^2 V$ then $\chi_W = \binom{X_1}{2} - X_2$, since $\sigma \in S_n$ fixes those basis elements $x_i \wedge x_j$ for which the cycle decomposition of σ contains the pair of fixed points $(i)(j)$, while $\sigma(x_i \wedge x_j) = -x_i \wedge x_j$ when the cycle decomposition of σ contains the 2-cycle $(i j)$.

One notable feature of these two examples is that the same polynomial describes the character of an entire family of similarly-defined S_n -representations, one for each $n \geq 1$. Of course, the expression of a class function on S_n as a character polynomial is not unique; for example X_N vanishes identically on S_n for all $N > n$. However, two character polynomials that agree on S_n for *infinitely many* n must be equal, so for a sequence of class functions χ_n on S_n it makes sense to ask about “the” character polynomial, if any, that realizes χ_n .

Definition 1.5 (Eventually polynomial characters). We say that a sequence χ_n of characters of S_n is *eventually polynomial* if there exist integers r, N and a polynomial $P(X_1, \dots, X_r)$ such that

$$\chi_n(\sigma) = P(X_1, \dots, X_r)(\sigma) \quad \text{for all } n \geq N \text{ and all } \sigma \in S_n. \quad (1)$$

The *degree* of the character polynomial $P(X_1, \dots, X_r)$ is defined by setting $\deg(X_i) = i$.

One of the most striking properties of FI-modules in characteristic 0 is that we have such a uniform description for the characters of any finitely generated FI-module.

Theorem 1.6 (Polynomiality of characters). *Let V be an FI-module over a field of characteristic 0. If V is finitely generated then the sequence of characters χ_{V_n} of the S_n -representations V_n is eventually polynomial.*

The *character polynomial* of a finitely generated FI-module is the polynomial $P(X_1, \dots, X_r)$ which gives the characters χ_{V_n} . It will always be integer-valued, meaning that $P(X_1, \dots, X_r) \in \mathbb{Z}$ whenever $X_1, \dots, X_r \in \mathbb{Z}$. In situations of interest one can typically produce an explicit upper bound on the degree of the character polynomial by computing the *weight* of V as in Definition 2.50. In particular this gives an upper bound for the number of variables r . Moreover, computing the *stability degree* of V (§2.4) gives explicit bounds on the range $n \geq N$ where χ_{V_n} is given by the character polynomial. This converts the problem of finding all the characters χ_{V_n} into a concrete finite computation. Note that Theorem 1.3 follows from Theorem 1.6, since

$$\dim V_n = \chi_{V_n}(\text{id}) = P(n, 0, \dots, 0).$$

One consequence of Theorem 1.6 is that χ_{V_n} only depends on “short cycles”, i.e. on cycles of length $\leq r$. This is a highly restrictive condition when n is much larger than r . For example, the proportion of permutations in S_n that have *no* cycles of length r or less is bounded away from 0 as n grows, and χ_{V_n} is constrained to be *constant* on this positive-density subset of S_n .

Little is known about the characters of the S_n -representations in Table 1; indeed in many cases closed form computations are out of reach. However, since every one of the sequences (1)–(9) comes from a finitely generated FI-module, Theorem 1.6 applies to each sequence.

Corollary 1.7. *The characters χ_{V_n} of each of the sequences (1)–(9) of S_n -representations in Table 1 are eventually polynomial.*

This result suggests that FI-modules can provide a powerful tool in situations where explicit information is unavailable.

FI \sharp -modules. It is often the case that FI-modules arising in nature carry an even more rigid structure. An FI \sharp -module is a functor from the category of *partial injections* of finite sets to modules over k (see §2.3). In contrast with the category of FI-modules, the category of FI \sharp -modules is close to being semisimple (see Theorem 2.24 for a precise statement). For FI \sharp -modules we improve Theorem 1.3 to the following very strong condition, which holds over fields of arbitrary characteristic.

Theorem 1.8. *Let V be an FI \sharp -module over any field k . The following are equivalent:*

1. $\dim(V_n)$ is bounded above by a polynomial in n .
2. $\dim(V_n)$ is exactly equal to a polynomial in n for all $n \geq 0$.

The power of Theorem 1.8 comes from the fact that in practice it is quite easy to prove that $\dim(V_n)$ is bounded above by a polynomial. This theorem can be extended to FI \sharp -modules over \mathbb{Z} (and even more general rings). When k is a field of characteristic 0, we strengthen Theorem 1.6 to show that the character of V_n is given by a single character polynomial for all $n \geq 0$.

We now focus in greater detail on two of the most striking applications of our results. Many other applications are given in Sections 3, 4 and 5.

Cohomology of configuration spaces. In §4 we prove a number of new theorems about configuration spaces on manifolds. Let $\text{Conf}_n(M)$ denote the configuration space of ordered n -tuples of distinct points in a space M :

$$\text{Conf}_n(M) := \{(p_1, \dots, p_n) \in M^n \mid p_i \neq p_j\}$$

Configuration spaces and their cohomology are of wide interest in topology and in algebraic geometry; for a sampling, see Fulton-MacPherson [FMac], McDuff [McD], or Segal [Se].

An injection $f: \{1, \dots, m\} \hookrightarrow \{1, \dots, n\}$ induces a map $\text{Conf}_n(M) \rightarrow \text{Conf}_m(M)$ sending (p_1, \dots, p_n) to $(p_{f(1)}, \dots, p_{f(m)})$. This defines a contravariant functor $\text{Conf}(M)$ from FI to the category of topological spaces. Thus for any fixed $i \geq 0$ and any ring k , we obtain an FI-module $H^i(\text{Conf}(M); k)$. Using work of Totaro [To] we prove that when M is a connected, oriented manifold of dimension ≥ 2 , the FI-module $H^i(\text{Conf}(M); \mathbb{Q})$ is finitely generated for each $i \geq 0$. Moreover, when $\dim M \geq 3$ we bound the weight and stability degree of this FI-module to prove the following.

Theorem 1.9. *If $\dim M \geq 3$, there is a character polynomial $P_{M,i}$ of degree i so that*

$$\chi_{H^i(\text{Conf}_n(M); \mathbb{Q})}(\sigma) = P_{M,i}(\sigma) \quad \text{for all } n \geq 2i \text{ and all } \sigma \in S_n.$$

In particular, the Betti number $b_i(\text{Conf}_n(M))$ agrees with a polynomial of degree i for all $n \geq 2i$.

If M is an open manifold we prove that $H^i(\text{Conf}(M); k)$ is in fact an $\text{FI}\sharp$ -module for any ring k . This implies that the character polynomial $P_{M,i}$ from Theorem 1.9 agrees with the character of $H^i(\text{Conf}_n(M); \mathbb{Q})$ for all $n \geq 0$. It also implies sharp constraints on the integral and mod- p cohomology of $\text{Conf}_n(M)$.

Theorem 1.10. *If M is an open manifold, each of the following invariants of $\text{Conf}_n(M)$ is given by a polynomial in n for all $n \geq 0$ (of degree i if $\dim M \geq 3$, and of degree $2i$ if $\dim M = 2$):*

1. *the i -th rational Betti number $b_i(\text{Conf}_n(M))$;*
2. *the i -th mod- p Betti number of $\text{Conf}_n(M)$;*
3. *the rank of $H^i(\text{Conf}_n(M); \mathbb{Z})$;*
4. *the rank of the p -torsion part of $H^i(\text{Conf}_n(M); \mathbb{Z})$.*

We believe that each of these results is new. Our theory also yields a new proof of [Ch, Theorem 1], which was used in [Ch] to give the first proof of rational homological stability for unordered configuration spaces of arbitrary manifolds. Our proof here is in a sense parallel to that of [Ch], but the new framework simplifies the mechanics of the proof considerably, and allows us to sharpen the bounds on the stable range. When $\dim(M) > 2$ we also prove homological stability for various configuration spaces of sets of “colored points”; the stable range here is better than that obtained for the same problem in [Ch, Theorem 5].

As a simple illustration of the above results, we obtain that the character of the S_n -representation $H^2(\text{Conf}(\mathbb{R}^2); \mathbb{Q})$ is given for all $n \geq 0$ by the single character polynomial

$$\chi_{H^2(\text{Conf}_n(\mathbb{R}^2); \mathbb{Q})} = 2 \binom{X_1}{3} + 3 \binom{X_1}{4} + \binom{X_1}{2} X_2 - \binom{X_2}{2} - X_3 - X_4. \quad (2)$$

Diagonal coinvariant algebras. In §3.2 we obtain new results about a well-studied object in algebraic combinatorics: the multivariate diagonal coinvariant algebra. The story begins in classical invariant theory. Let k be a field of characteristic 0. Fix $r \geq 1$. For each $n \geq 0$ we consider the algebra of polynomials

$$k[\mathbf{X}^{(r)}(n)] := k[x_1^{(1)}, \dots, x_n^{(1)}, \dots, x_1^{(r)}, \dots, x_n^{(r)}]$$

in r sets of n variables. The symmetric group S_n acts on this algebra via the diagonal action:

$$\sigma \cdot x_j^{(i)} := x_{\sigma(j)}^{(i)}$$

Chevalley and Weyl computed the S_n -invariants of this action, the so-called *multisymmetric polynomials* (see [Weyl, II.A.3]). Let I_n be the ideal in $k[\mathbf{X}^{(r)}(n)]$ generated by the multisymmetric polynomials with vanishing constant term. The *r -diagonal coinvariant algebra* is defined to be the k -algebra

$$R^{(r)}(n) := k[\mathbf{X}^{(r)}(n)]/I_n.$$

Each coinvariant algebra $R^{(r)}(n)$ is known to be a finite-dimensional S_n -representation. These representations have been objects of intense study in algebraic combinatorics. Borel proved that $R^{(1)}(n)$ is isomorphic as an S_n -representation to the cohomology $H^*(\text{GL}_n \mathbb{C}/B; k)$ of the complete flag variety $\text{GL}_n \mathbb{C}/B$. Furthermore Chevalley [Che, Theorem B] proved that $R^{(1)}(n)$ is isomorphic to the regular representation of S_n , so $\dim R^{(1)}(n) = n!$. Haiman [Hai] gave a geometric interpretation for $R^{(2)}(n)$ and used it to prove the “ $(n+1)^{n-1}$ Conjecture”, which states that

$$\dim(R^{(2)}(n)) = (n+1)^{n-1}.$$

For $r > 2$ the dimension of $R^{(r)}(n)$ is not known.

The polynomial algebra $k[\mathbf{X}^{(r)}(n)]$ naturally has an r -fold multi-grading, where a monomial has multi-grading $J = (j_1, \dots, j_r)$ if its total degree in the variables $x_1^{(k)}, \dots, x_n^{(k)}$ is j_k . This multi-grading is S_n -invariant, and descends to an S_n -invariant multi-grading

$$R^{(r)}(n) = \bigoplus_J R_J^{(r)}(n)$$

on the r -diagonal coinvariant algebra $R^{(r)}(n)$. It is a well-known problem to describe these graded pieces as S_n -representations.

Problem 1.11. *For each $r \geq 2, n \geq 1$, and $J = (j_1, \dots, j_r)$, compute the character of each $R_J^{(r)}(n)$ as an S_n -representation, at least for n sufficiently large. In particular, find a formula for $\dim(R_J^{(r)}(n))$.*

For $r = 1$ this problem was solved independently by Stanley, Lusztig, and Kraskiewicz–Weyman (see [CF, §7.1]). For $r = 2$ Haiman [Hai2] gave a formula for these characters in terms of Macdonald polynomials and a “rather mysterious operator” (see [HHLRU] for a discussion). In the lowest degree cases the computation is elementary. For example, it is easy to check that:

$$\begin{aligned} \dim R_1^{(1)}(n) &= n - 1 & \chi_{R_1^{(1)}(n)} &= X_1 - 1 & \text{for } n \geq 1 \\ \dim R_2^{(1)}(n) &= \binom{n}{2} - 1 & \chi_{R_2^{(1)}(n)} &= \binom{X_1}{2} + X_2 - 1 & \text{for } n \geq 2 \\ \dim R_{11}^{(2)}(n) &= 2\binom{n}{2} - n & \chi_{R_{11}^{(2)}(n)} &= 2\binom{X_1}{2} - X_1 & \text{for } n \geq 2 \end{aligned} \tag{3}$$

Note that in these low-degree cases, once n is sufficiently large the dimension of $R_J^{(r)}(n)$ is polynomial in n , as is its character. For $r > 2$ it seems that almost nothing is known about Problem 1.11, except in low-degree cases (see [Be, §4] for a discussion of these cases). However, the descriptions in (3) are just the simplest examples of a completely general phenomenon.

Theorem 1.12 (Characters of $R_J^{(r)}(n)$ are eventually polynomial in n). *For any fixed $r \geq 1$ and $J = (j_1, \dots, j_r)$, the characters $\chi_{R_J^{(r)}(n)}$ are eventually polynomial in n of degree at most $|J|$. In particular there exists a polynomial $P_J^{(r)}(n)$ of degree at most $|J|$ such that*

$$\dim(R_J^{(r)}(n)) = P_J^{(r)}(n) \text{ for all } n \gg 0.$$

Indeed, we prove that (the dual of) $R_J^{(r)}$ is a finitely generated FI-module, and thus deduce the theorem from Theorem 1.6. It is clear *a priori* that $\dim(R_J^{(r)}(n))$ grows no *faster* than $O(n^{|J|})$, but the fact that this dimension eventually coincides exactly with a polynomial is new, as is the polynomial behavior of the character of $R_J^{(r)}(n)$. We emphasize that beyond this bound on the degree, Theorem 1.12 gives no information on the polynomials $P_J^{(r)}(n)$.

Problem 1.13 (Character polynomials for diagonal coinvariant algebras). *Describe the polynomials $P_J^{(r)}(n)$ whose existence is guaranteed by Theorem 1.12.*

Murnaghan’s theorem. Given a field of characteristic 0 and a partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$, for any $n \geq |\lambda| + \lambda_1$ we define $V(\lambda)_n$ to be the irreducible representation of S_n corresponding to the partition

$$\lambda[n] := (n - |\lambda|, \lambda_1, \dots, \lambda_\ell).$$

Murnaghan's theorem states that, for any partitions λ and μ , there are coefficients $g_{\mu,\lambda}^\nu$ such that the tensor product $V(\lambda)_n \otimes V(\mu)_n$ decomposes into S_n -irreducibles as

$$V(\lambda)_n \otimes V(\mu)_n = \bigoplus_{\nu} g_{\lambda,\mu}^\nu V(\nu)_n$$

for all sufficiently large n . The first complete proof of this theorem was given by Littlewood [Li] in 1957, but the coefficients $g_{\lambda,\mu}^\nu$ remain unknown in general. We will show in §2.8 that Murnaghan's theorem is an easy consequence of the fact that the tensor product of finitely generated FI-modules is finitely generated. From this point of view, Murnaghan's theorem is not merely an assertion about a list of numbers, but becomes a structural statement about the single mathematical object $V(\lambda) \otimes V(\mu)$.

Connection with representation stability. In [CF], Church and Farb introduced the theory of *representation stability*. Stability theorems in topology and algebra typically assert that in a given sequence of vector spaces with linear maps

$$\cdots \rightarrow V_n \rightarrow V_{n+1} \rightarrow V_{n+2} \rightarrow \cdots$$

the maps $V_n \rightarrow V_{n+1}$ are isomorphisms for n large enough. The goal of representation stability is to provide a framework for generalizing these results to situations when each vector space V_n has an action of the symmetric group S_n (or other natural families of groups). It is clearly wrong to ask that V_n and V_{n+1} be isomorphic, since they are representations of different groups. Representation stability provides a formal way of saying that the “names” of the S_n -representations V_n stabilize, and a language to describe this stabilization rigorously; see §2.6 below for the precise definition. Representation stability has been proved in many cases, and gives new conjectures in others; see [CF] and [Ch], as well as [J1, Wil]. An FI-module bundles an entire sequence of S_n -representations into a single object. In this framework, representation stability becomes finite generation of the FI-module.

Theorem 1.14 (Finite generation vs. representation stability). *An FI-module V over a field of characteristic 0 is finitely generated if and only if the sequence $\{V_n\}$ of S_n -representations is uniformly representation stable in the sense of [CF] and each V_n is finite-dimensional. In particular, for any finitely generated FI-module V , we have for sufficiently large n a decomposition:*

$$V_n \simeq \bigoplus c_\lambda V(\lambda)_n$$

where the coefficients c_λ do not depend on n .

In the language of [CF], the new result here is that “surjectivity” implies “uniform multiplicity stability” for FI-modules. This turns out to be very useful, because in practice finite generation is much easier to prove than representation stability. A key ingredient in this theorem is the “monotonicity” proved by the first author in [Ch, Theorem 2.8], and as a byproduct of the proof we obtain that finitely generated FI-modules are monotone in this sense. Once again, the stability degree defined in Section 2.4 allows us, in many cases of interest, to replace “sufficiently large n ” with an explicit range.

Our new point of view also simplifies the description of many representation-stable sequences by encoding them as a single object. As a simple example, in [CF] we showed that

$$H^2(\text{Conf}_n(\mathbb{R}^2); \mathbb{Q}) = V(1)_n^{\oplus 2} \oplus V(1, 1)_n^{\oplus 2} \oplus V(2)_n^{\oplus 2} \oplus V(2, 1)_n^{\oplus 2} \oplus V(3)_n \oplus V(3, 1)_n$$

for all $n \geq 7$, and separate descriptions were necessary for smaller n . In the language of the present paper we can simply write

$$H^2(\text{Conf}(\mathbb{R}^2); \mathbb{Q}) = M(\begin{smallmatrix} \square \\ \square \end{smallmatrix}) \oplus M(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}) \quad (4)$$

to simultaneously describe $H^2(\mathrm{Conf}_n(\mathbb{R}^2); \mathbb{Q})$ for *all* $n \geq 0$ (see §2.1 for notation, and §3.1 for the proof). One appealing feature here is that the “instability” of the sequence $H^2(\mathrm{Conf}_n(\mathbb{R}^2); \mathbb{Q})$, meaning the qualitatively different descriptions that are necessary for $n < 7$, is already encoded in the right-hand side of (4). Based on computer calculations, John Wiltshire-Gordon (personal communication) has formulated a precise conjecture for the decomposition of $H^i(\mathrm{Conf}(\mathbb{R}^2); \mathbb{Q})$ as in (4) for all $i \geq 0$. Our results give a decomposition as in (4) with \mathbb{R}^2 replaced by any open manifold, but we do not in general know the right-hand side explicitly.

FI-modules for other groups. The theory of representation stability developed in [CF] applies not only to representations of S_n , but also to families such as $\mathrm{GL}_n \mathbb{Q}$, $\mathrm{Sp}_{2n} \mathbb{Q}$, and hyperoctahedral groups. Similarly, we extend the framework of FI-modules to algebraic and arithmetic groups in [CEF], and Wilson extends it to all classical families of Weyl groups in [Wi2].

Relation with other work. We record here some areas of overlap between the material in this paper and the work of others.

- A recent paper of Putman [Pu] proposes an alternative representation stability condition called “central stability” and proves that it holds for the mod- p cohomology of congruence groups. In the language of FI-modules, central stability turns out to be equivalent to a finite presentation condition (“presented in finite degree” in Definition 2.48). We prove in this paper that finitely generated FI-modules in characteristic 0 are also finitely presented. Thus for FI-modules in characteristic 0, finite generation, representation stability, and central stability are equivalent.
- The category of FI-modules, along with related abelian categories, has also been considered by researchers in the field of polynomial functors. See, e.g., the recent work of Djament and Vespa [DV] which studies the stable behavior of the cohomology groups $H^i(S_n, V_n)$ for FI-modules V , or the paper of Helmstutler [He] whose general model-theoretic framework can be understood to include the construction and some of the properties of FI \sharp -modules.
- While this paper was being prepared, we learned about the recent work of Andrew Snowden in [Sn], which has some overlap with our own. In particular, Snowden proves in [Sn, Theorem 2.3] (albeit in quite different language) that the category of FI-modules over a field of characteristic 0 is Noetherian (see Theorem 2.60). FI-modules can be viewed as modules for the “exponential” twisted commutative algebra (see e.g. [AM]), and Snowden in fact proves this Noetherian property for a more general class of twisted commutative algebras. Applying this perspective to FI \sharp -modules yields examples of (divided power) D -modules both in characteristic 0 and in positive characteristic, as we explain in [CEF]. After this paper was released, Sam–Snowden gave in [SS] a more detailed analysis of the algebraic structure of the category of FI-modules in characteristic 0.
- Objects in Deligne’s category $\mathrm{Rep}(S_t)$, where t is a complex number, are closely related to sequences of S_n -representations whose characters are eventually polynomial (see Deligne [De], Knop [Kn], or Etingof’s lecture [Et]). Is there a sense in which a finite-type object of $\mathrm{Rep}(S_t)$ can be specialized to a finitely-generated FI-module? An interesting example is provided by the recent work of Ren and Schedler [RS] on spaces of invariant differential operators on symplectic manifolds. Their results are consistent with the proposition that their sequence $\mathrm{Inv}_n(V)$ carries the structure of a finitely generated FI-module. Does it?

Outline of the paper. In Section 2 we lay out the definitions and prove the foundational properties of FI-modules. The remainder of the paper consists of applications of these results.

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2 FI-modules: definitions and properties

In this section we develop the basic theory of FI-modules, including definitions and foundational properties. We also introduce the closely related category of FI \sharp -modules, which are significantly more rigid and well-behaved than FI-modules, especially for coefficient rings other than fields of characteristic 0.

2.1 Basic properties and examples of FI-modules

Notational conventions. Fix a commutative ring k . The case when k is a field will be foremost in our minds, and our notation is chosen accordingly. For example, by a *representation over k of a group G* we mean a k -module V together with an action of G on V by k -module automorphisms, or, in other words, a module for the group ring kG . By the *dual representation* V^* we mean the k -module $\text{Hom}_k(V, k)$ together with its induced G -action (we will mainly use this notion when k is a field). Given a finite group G with subgroup H , if V is a representation of H we write $\text{Ind}_H^G V$ for the “induced representation” $V \otimes_{kH} kG$. Given two groups G and H , if V is a representation of G , then $V \boxtimes k$ denotes the same k -module V , interpreted as a representation of $G \times H$ on which H acts trivially.

Recall from Definition 1.1 that FI denotes the category of **f**inite sets and **i**njections. Let FI-Mod be the category of FI-modules over k ; that is, FI-Mod is the category of functors from FI to the category $k\text{-Mod}$ of k -modules. If V is an FI-module, we write V_n for $V(\mathbf{n})$. Similarly, if $f: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ is an injection, we often write $f_*: V_m \rightarrow V_n$ for the map $V(f)$.

It is well-known (see, e.g., [Wei, A.4.3]) that the category of functors from any small category to an abelian category is abelian, so FI-Mod is an abelian category. Moreover, notions such as kernel, cokernel, subobject, quotient object, injection, or surjection are all defined “pointwise”, meaning that a property holds for an FI-module V if and only if it holds for each V_n . For example, we say a map $V \rightarrow W$ of FI-modules is a surjection if and only if the maps $V_n \rightarrow W_n$ are surjections for all n .

Consistent sequences vs. FI-modules. Recall from [CF] that a sequence $\{V_n\}$ of finite-dimensional S_n -representations equipped with linear maps $\phi_n: V_n \rightarrow V_{n+1}$ is called *consistent* if the following diagram commutes for each n and each $\sigma \in S_n$:

$$\begin{array}{ccc} V_n & \xrightarrow{\phi_n} & V_{n+1} \\ \sigma \downarrow & & \downarrow \iota(\sigma) \\ V_n & \xrightarrow{\phi_n} & V_{n+1} \end{array}$$

Here $\iota: S_n \rightarrow S_{n+1}$ is the standard inclusion. Given an FI-module V , we obtain a consistent sequence of S_n -representations by taking the S_n -representations V_n making up the FI-module V , together with the maps ϕ_n obtained by applying V to the standard inclusions $I_n: \{1, \dots, n\} \hookrightarrow \{1, \dots, n+1\}$ defined by $I_n(i) = i$. But FI-modules are much more restrictive than consistent sequences. In fact, we can describe exactly the difference between a general consistent sequence and an FI-module.

Lemma 2.1. *A consistent sequence $\{V_n, \phi_n\}$ can be promoted (in a unique way) to an FI-module if and only if it satisfies the following condition for all $m \leq n$: Given two permutations $\sigma, \sigma' \in S_n$, and an element $v \in V_n$ lying in the image of V_m , we have*

$$\sigma|_{\{1, \dots, m\}} = \sigma'|_{\{1, \dots, m\}} \implies \sigma(v) = \sigma'(v). \quad (5)$$

Remark 2.2. It is easy to construct examples of consistent sequences which violate this condition. For example, the sequence of regular representations

$$k \xrightarrow{\phi_0} k \xrightarrow{\phi_1} k[S_2] \xrightarrow{\phi_2} k[S_3] \rightarrow \dots \quad (6)$$

induced by the standard inclusions $S_m \hookrightarrow S_{m+1}$ is a consistent sequence of representations. However it is easy to check that this sequence does not satisfy (5), and thus cannot be extended to an FI-module.

Proof of Lemma 2.1. Let $I_{m,n}: \{1, \dots, m\} \hookrightarrow \{1, \dots, n\}$ be the standard inclusion. Note that $\sigma|_{\{1, \dots, m\}} = \sigma'|_{\{1, \dots, m\}}$ if and only if $\sigma \circ I_{m,n} = \sigma' \circ I_{m,n}$.

We are given maps $\phi_n: V_n \rightarrow V_{n+1}$; for any $m < n$ let $\phi_{n,m} := \phi_{n-1} \circ \dots \circ \phi_{m+1} \circ \phi_m$. Let $f: \{1, \dots, m\} \hookrightarrow \{1, \dots, n\}$ be an injection; we seek to define a map $f_*: V_m \rightarrow V_n$. For any such f we can write $f = \sigma \circ I_{m,n}$ for some $\sigma \in S_n$. We define f_* to be $\sigma_* \circ \phi_{m,n}$; this is well defined if and only if the condition (5) holds, and uniqueness is immediate.

It remains to check that $(f \circ g)_* = f_* \circ g_*$ for $g: \{1, \dots, i\} \hookrightarrow \{1, \dots, m\}$. Write $f = \sigma_f \circ I_{m,n}$ and $g = \sigma_g \circ I_{i,m}$ for $\sigma_f \in S_n$ and $\sigma_g \in S_m$. Note that $I_{m,n} \circ \sigma_g = \widetilde{\sigma_g} \circ I_{m,n}$ where $\widetilde{\sigma_g} \in S_n$ is the image of σ_g under the standard inclusion $S_m \hookrightarrow S_n$. Thus we have $f \circ g = \sigma_f \circ \widetilde{\sigma_g} \circ I_{i,n}$. The consistency of the sequence V_n implies that $(\widetilde{\sigma_g})_* \circ \phi_{m,n} = \phi_{m,n} \circ (\sigma_g)_*$. We conclude that

$$f_* \circ g_* = (\sigma_f)_* \circ \phi_{m,n} \circ (\sigma_g)_* \circ \phi_{i,m} = (\sigma_f)_* \circ (\widetilde{\sigma_g})_* \circ \phi_{i,n} = (f \circ g)_*$$

as desired.

Finally, it is easy to check that the condition is necessary, since if v lies in the image of V_m , we can write $v = \phi_{m,n}(w) = (I_{m,n})_*(w)$. Since the assumption implies that $\sigma \circ I_{m,n} = \sigma' \circ I_{m,n}$, we have $\sigma(v) = \sigma'(v)$ as desired. \square

Sources of FI-modules. The class of FI-modules is evidently closed under application of any covariant functorial construction on k -modules. In particular, tensor products of FI-modules are FI-modules, as are symmetric products and exterior products (when k is not a field, we must fix some functorial definition of these constructions). The *dual* of an FI-module, on the other hand, is not naturally an FI-module.

Definition 2.3 (co-FI-modules). Let co-FI denote the *opposite category* of FI; that is, co-FI has the same objects as FI, and the morphisms in co-FI from \mathbf{m} to \mathbf{n} are the morphisms in FI from \mathbf{n} to \mathbf{m} . A *co-FI-module* over k is a functor from co-FI to k -Mod. If V is an FI-module, its dual V^* is naturally a co-FI-module.

An extremely useful source of FI-modules is a collection of spaces X_n with S_n -actions and appropriate maps $X_m \rightarrow X_n$ between them.

Definition 2.4 (FI-spaces and co-FI-spaces). An *FI-space* X is a functor from FI to the category Top of topological spaces. These have been considered elsewhere in the topological literature under the names “ \mathcal{I} -spaces” or “ Λ -spaces”; see e.g. [CMT]. Composing with a homology functor $H_i(-; k)$ yields an FI-module which we call $H_i(X; k)$. Similarly, a co-FI-space X is a functor $X: \text{co-FI} \rightarrow \text{Top}$, and in this case the cohomology $H^i(X; k)$ forms an FI-module.

An *FI-space* (resp. *co-FI-space*) up to homotopy is a functor from FI (resp. co-FI) to the homotopy category of topological spaces \mathbf{hTop} . Concretely, this means that for each n we have a space X_n , for each injection $\{1, \dots, m\} \hookrightarrow \{1, \dots, n\}$ we have a map $X_m \rightarrow X_n$, and the corresponding diagrams all commute up to homotopy. Since homotopy classes of maps induce well-defined maps on homology, the homology groups $H_i(X; k)$ (resp. cohomology groups $H^i(X; k)$) again form an FI-module in this case.

Free FI-modules. In the remainder of this section we define certain families of FI-modules that can be thought of as the FI-modules “freely generated” by an S_a -representation.

Definition 2.5 (The FI-module $M(\mathbf{m})$). For any $m \geq 0$, let $M(m)$ be the FI-module that assigns to a finite set S the free k -module spanned by $\mathrm{Hom}_{\mathrm{FI}}(\mathbf{m}, S)$, so $M(m)_n$ has basis indexed by the injections $\{1, \dots, m\} \hookrightarrow \{1, \dots, n\}$ — in other words, by ordered sequences of m distinct elements of $\{1, \dots, n\}$. Of particular interest is $M(0)$, which in degree n is $M(0)_n \simeq k$, the 1-dimensional trivial representation of S_n , and $M(1)$, which in degree n is $M(1)_n \simeq k^n$, the permutation representation of S_n . The symmetric group S_m acts on the FI-module $M(m)$ by precomposing $\mathrm{Hom}_{\mathrm{FI}}(\mathbf{m}, S)$ with $\mathrm{Hom}_{\mathrm{FI}}(\mathbf{m}, \mathbf{m}) = S_m$.

Let $S_a\text{-Rep}$ denote the category of S_a -representations over k , i.e. the category of $k[S_a]$ -modules. For each integer $a \geq 0$ there is a natural forgetful functor

$$\pi_a: \mathrm{FI}\text{-Mod} \rightarrow S_a\text{-Rep}$$

defined by $\pi_a(V) = V_a$. Note that π_a is an exact functor.

Proposition 2.6. *For each $a \geq 0$ the map π_a has a left adjoint $\mu_a: S_a\text{-Rep} \rightarrow \mathrm{FI}\text{-Mod}$ given by $\mu_a(W) = W \otimes_{k[S_a]} M(a)$. Explicitly, we have*

$$(\mu_a(W))_n = \begin{cases} 0 & n < a \\ \mathrm{Ind}_{S_a \times S_{n-a}}^{S_n} W \boxtimes k & n \geq a \end{cases} \quad (7)$$

where \boxtimes denotes the external tensor product.

Proof. A basis for $M(a)_n$ is by definition given by $\mathrm{Hom}_{\mathrm{FI}}(\mathbf{a}, \mathbf{n})$, so $M(a)_n$ can be identified with $k[S_n/S_{n-a}] \simeq \mathrm{Ind}_{S_a \times S_{n-a}}^{S_n} k[S_a] \boxtimes k$. This yields the formula above for $(\mu_a(W))_n$. It remains to show that μ_a is left adjoint to π_a .

Let V be an FI-module. We need to show that $\mathrm{Hom}_{\mathrm{FI}\text{-Mod}}(\mu_a(W), V) \simeq \mathrm{Hom}_{S_a\text{-Rep}}(W, V_a)$. First, consider the case when $W = k[S_a]$, so that $\mu_a(W) = M(a)$. A homomorphism of FI-modules $M(a) \rightarrow V$ is determined by the image of the identity $I_{a,a} \in \mathrm{Hom}(\mathbf{a}, \mathbf{a}) \subset M(a)_a$, since every injection $f \in \mathrm{Hom}(\mathbf{a}, \mathbf{n})$ can be written as $f \circ I_{a,a} = f_* I_{a,a}$. Thus

$$\mathrm{Hom}_{\mathrm{FI}\text{-Mod}}(M(a), V) \simeq V_a \simeq \mathrm{Hom}_{S_a\text{-Rep}}(k[S_a], V_a)$$

as desired. We deduce the claim for arbitrary W as follows:

$$\mathrm{Hom}_{\mathrm{FI}\text{-Mod}}(W \otimes M(a), V) \simeq \mathrm{Hom}_{S_a\text{-Rep}}(W, \mathrm{Hom}_{\mathrm{FI}\text{-Mod}}(M(a), V)) \simeq \mathrm{Hom}_{S_a\text{-Rep}}(W, V_a) \quad \square$$

Definition 2.7 (The FI-module $M(W)$). We combine the functors μ_a into a single functor $M(-): \bigoplus_a S_a\text{-Rep} \rightarrow \mathrm{FI}\text{-Mod}$ by

$$M\left(\bigoplus W_a\right) := \bigoplus \mu_a(W_a),$$

where W_a is a $k[S_a]$ -module. By slight abuse of notation, for a single $k[S_a]$ -module W_a we still write $M(W_a) := \mu_a(W_a)$. In particular, $M(a)$ is another name for $M(k[S_a])$.

When k is a field, the dimension of the vector space $M(W)_n$ can be computed exactly:

$$\dim M(W)_n = \sum_{a \geq 0} \dim(W_a) \cdot \binom{n}{a} \quad (8)$$

In particular, the dimension of $M(W)_n$ is given for all $n \geq 0$ by a single polynomial in n ; we will see in Theorem 2.67 that the same holds for its character.

Irreducible representations of S_n in characteristic 0. Recall that the irreducible representations of S_n over any field k of characteristic 0 are classified by the partitions λ of n . A *partition* of n is a sequence $\lambda = (\lambda_1 \geq \dots \geq \lambda_\ell > 0)$ with $\lambda_1 + \dots + \lambda_\ell = n$; we write $|\lambda| = n$ or $\lambda \vdash n$. We will denote by V_λ the irreducible representation corresponding to the partition λ . The representation V_λ can be obtained as the image $k[S_n] \cdot c_\lambda$ of a certain idempotent c_λ (the “Young symmetrizer”) in the group algebra $k[S_n]$. Every irreducible representation of S_n is defined over \mathbb{Q} , and as a result the decomposition of an S_n -representation over a field k of characteristic 0 does not depend on the field k . Every representation of S_n is self-dual.

Definition 2.8 (The irreducible representation $V(\lambda)_n$). If λ is any partition, then for any $n \geq |\lambda| + \lambda_1$ we define the *padded partition*

$$\lambda[n] := (n - |\lambda|, \lambda_1, \dots, \lambda_\ell).$$

For $n \geq |\lambda| + \lambda_1$, we define $V(\lambda)_n$ to be the irreducible S_n -representation

$$V(\lambda)_n := V_{\lambda[n]}.$$

Every irreducible representation of S_n is of the form $V(\lambda)_n$ for a unique partition λ . We sometimes replace λ by its corresponding Young diagram. In this notation, the trivial representation of S_n is $V(0)_n$ and the standard representation is $V(\square)_n$ for all $n \geq 2$. The usual linear algebra operations behave well with respect to this notation. For example, the identity

$$\wedge^3 V(\square)_n \simeq V\left(\begin{array}{|c|} \hline \square \\ \hline \end{array}\right)_n$$

holds whenever both sides are defined, namely whenever $n \geq 4$.

Definition 2.9 (The FI-module $M(\lambda)$). When k is a field of characteristic 0 and λ is a partition of a , we denote by $M(\lambda)$ the FI-module $M(V_\lambda) = \mu_a(V_\lambda)$.

Many natural combinatorial constructions correspond to FI-modules of this form. For example, the FI-module $M(\square\square\square)$ is the linearization of the functor $S \mapsto \binom{S}{3}$, since a basis for $M(\square\square\square)_n$ is given by the 3-element subsets of $\{1, \dots, n\}$. Similarly, the FI-module $M(\begin{array}{|c|} \hline \square \\ \hline \end{array})$ is the linearization of the functor sending S to the collection of oriented edges $x \rightarrow y$ between distinct elements of S (oriented in the sense that $y \rightarrow x$ is the negative of $x \rightarrow y$).

Remark 2.10 (FI-Mod is not semisimple). In some sense the category of FI-modules might be thought of as a “limit” of the category of representations of S_n as $n \rightarrow \infty$. But care is necessary. For example, the category of FI-modules over k is not semisimple, even when k is a field of characteristic 0.

For one example of the failure of semisimplicity, let V be the FI-module with $V_0 = k$ and $V_i = 0$ for all $i > 1$. There is an obvious surjection $M(0) \twoheadrightarrow V$, but this surjection is not split: if there were a section $s: V \rightarrow M(0)$, the image $s(V_0)$ would be a nonzero submodule of $M(0)_0 \simeq k$ which is sent to zero in $M(0)_1 \simeq k$, and there is no such submodule.

Example 2.11 (The FI-space $\Delta^{\bullet-1}$). The symmetric group S_n acts on the $(n-1)$ -simplex Δ^{n-1} by permuting its n vertices. These simplices together form an FI-space $\Delta^{\bullet-1}$ whose n -th space is Δ^{n-1} . Concretely, we define the FI-simplicial complex $\Delta^{\bullet-1}$ by letting $\Delta^{\bullet-1}(S)$ be the full simplicial complex $(S, 2^S)$ with vertex set S . An injection $S \hookrightarrow T$ induces a simplicial map $(S, 2^S) \hookrightarrow (T, 2^T)$.

The cellular chains of $\Delta^{\bullet-1}$ provide a projective resolution for the FI-module V from Remark 2.10. Let $C_m(\Delta^{\bullet-1})$ denote the FI-module over k spanned by the m -simplices of $\Delta^{\bullet-1}$:

$$C_m(\Delta^{\bullet-1})(S) := C_m(\Delta^{\bullet-1}(S); k)$$

The reduced cellular homology of Δ^n is computed by the augmented chain complex of FI-modules

$$\cdots \rightarrow C_3(\Delta^{\bullet-1}) \rightarrow C_2(\Delta^{\bullet-1}) \rightarrow C_1(\Delta^{\bullet-1}) \rightarrow C_0(\Delta^{\bullet-1}) \rightarrow M(0) \rightarrow 0. \quad (9)$$

Since Δ^n is contractible for all $n \geq 0$, its reduced homology vanishes in every dimension, so the complex (9) is exact for all $n > 0$. The sole exception is when $n = 0$, when $\Delta^{n-1} = \Delta^{-1}$ is the empty set, and (9) is just the complex $\cdots \rightarrow 0 \rightarrow k \rightarrow 0$. Thus (9) provides a resolution of the FI-module V which has $V_0 = k$ and $V_n = 0$ for $n > 0$, as claimed.

We can identify the FI-modules $C_m(\Delta^{\bullet-1})$ explicitly. By definition $C_m(\Delta^{\bullet-1})(S)$ is the free k -module on the $(m+1)$ -element subsets of S , twisted by a sign corresponding to the orientation of the simplex. Thus $C_m(\Delta^{\bullet-1})$ is isomorphic as an FI-module to $M(\varepsilon_{m+1})$, where ε_n denotes the sign representation of S_n over k . We can thus identify (9) with the following resolution of V .

$$\cdots \rightarrow M\left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}\right) \rightarrow M\left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}\right) \rightarrow M\left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}\right) \rightarrow M(\square) \rightarrow M(0) \rightarrow V \rightarrow 0 \quad (10)$$

The above holds over an arbitrary ring k , except for a slight abuse of notation in (10) for $M(\varepsilon_n)$. Moreover if k contains \mathbb{Q} , then ε_n is a projective $k[S_n]$ -module. Since the left adjoint to any exact functor preserves projective objects [Wei, Proposition 2.3.10], $M(\varepsilon_n)$ is a projective FI-module. Thus if k contains \mathbb{Q} , (10) provides a projective resolution of the FI-module V (see also Theorem 2.29).

2.2 Generators for an FI-module

Definition 2.12 (Span). If V is an FI-module and Σ is a subset of the disjoint union $\coprod V_n$, we define the *span* $\text{span}_V(\Sigma)$ to be the minimal sub-FI-module of V containing each element of Σ . We say that $\text{span}_V(\Sigma)$ is the *sub-FI-module of V generated by Σ* . We sometimes write $\text{span}(\Sigma)$ when there is no ambiguity.

Remark 2.13. Given an FI-module V and an element $v \in V_m$, there is a natural map $M(m) \rightarrow V$ whose image is exactly $\text{span}_V(v)$. To see this, note that $\text{span}_V(v)_n$ is spanned by the images $f_*(v)$ as f ranges over $f \in \text{Hom}_{\text{FI}}(\mathbf{m}, \mathbf{n})$. Since $M(m)_n$ has a basis given by $\text{Hom}_{\text{FI}}(\mathbf{m}, \mathbf{n})$, we define our map by sending $f \in \text{Hom}_{\text{FI}}(\mathbf{m}, \mathbf{n})$ to $f_*(v) \in V_n$, and it is easy to check that this defines a map of FI-modules. More generally, if Σ is the disjoint union of $\Sigma_n \subset V_n$, there is a natural map $\bigoplus_{n \geq 0} M(n)^{\oplus \Sigma_n} \rightarrow V$ whose image is $\text{span}_V(\Sigma)$.

Definition 2.14 (Generation in degree $\leq m$). We say that an FI-module V is *generated in degree $\leq m$* if V is generated by elements of V_k for $k \leq m$. For simplicity we let $\text{span}(V_{\leq m})$ denote $\text{span}(\coprod_{k \leq m} V_k)$, and with this notation we have

$$V \text{ is generated in degree } \leq m \quad \Longleftrightarrow \quad \text{span}(V_{\leq m}) = V.$$

Definition 2.15 (Finite generation). We say that an FI-module V is *finitely generated* if there is a finite set of elements v_1, \dots, v_k with $v_i \in V_{n_i}$ which *generates* V , meaning that $\text{span}(v_1, \dots, v_k) = V$.

We will say that V is *finitely generated in degree $\leq m$* if V is generated in degree $\leq m$ and V is finitely generated. Note that this implies that there exists a finite generating set v_1, \dots, v_k with $v_i \in V_{m_i}$ for which $m_i \leq m$ for all i . Remark 2.13 implies the following characterization of finitely generated FI-modules in terms of the free FI-modules $M(m)$.

Proposition 2.16 (Finite generation in terms of $M(m)$). *An FI-module V is finitely generated if and only if it admits a surjection $\bigoplus_i M(m_i) \twoheadrightarrow V$ for some finite sequence of integers $\{m_i\}$.*

One consequence of Proposition 2.16 is that if V is a finitely generated FI-module then V_n is a finitely-generated k -module for each $n \geq 0$. Proposition 2.16 will allow us to reduce a number of finite generation problems to the corresponding problem for the particular FI-modules $M(m)$; see for example the proofs of Proposition 2.31 or Proposition 2.61 below. We record the following proposition, which is immediate from the definitions.

Proposition 2.17. *Let $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ be a short exact sequence of FI-modules. If V is generated in degree $\leq m$ (resp. finitely generated), then W is generated in degree $\leq m$ (resp. finitely generated). If both U and W are generated in degree $\leq m$ (resp. finitely generated), then V is generated in degree $\leq m$ (resp. finitely generated).*

We conclude this section with another perspective on generators for an FI-module V that will be used in later sections.

Definition 2.18 (The functor H_0). We define the functor $H_0: \text{FI-Mod} \rightarrow \bigoplus_{n \geq 0} S_n\text{-Rep}$ as follows. Given an FI-module V , the degree- n part of $H_0(V)$ is the S_n -representation

$$H_0(V)_n := V_n / \text{span}(V_{<n})_n,$$

where as above $\text{span}(V_{<n})$ is an abbreviation for $\text{span}(\bigsqcup_{k < n} V_k)$. Note that an FI-module V is generated in degree $\leq m$ if and only if $H_0(V)_n$ vanishes for all $n > m$. Similarly, V is finitely generated if and only if the underlying k -module $\bigoplus_{n \geq 0} H_0(V)_n$ is finitely generated.

If W is a $k[S_n]$ -representation, the FI-module $M(W)$ is generated by $M(W)_n \simeq W$, and it is easy to see that $H_0(M(W))$ consists just of W in degree n . For a general FI-module V , we would like to think of $H_0(V)$ as giving a “minimal generating set” for V , and ideally we would have a (non-canonical) surjection $M(H_0(V)) \twoheadrightarrow V$. This will not be possible for FI-modules over a general ring k , or even over a field of positive characteristic. However, we will show that such a surjection does exist when k is a field of characteristic 0 (Proposition 2.43) or when V has the additional structure of an FI \sharp -module (Theorem 2.24). In the latter case we in fact have an isomorphism $M(H_0(V)) \simeq V$.

2.3 FI \sharp -modules

In applications we will often encounter sequences of S_n -representations that simultaneously carry both an FI-module and a co-FI-module structure. Moreover these two structures are frequently compatible, in a sense we now make precise, and such objects are extremely rigid.

Definition 2.19 (FI \sharp -modules). Let $\text{FI}\sharp$ be the category whose objects are finite sets, and in which $\text{Hom}_{\text{FI}\sharp}(S, T)$ is the set of triples (A, B, ϕ) with A a subset of S , B a subset of T and $\phi: A \rightarrow B$ an isomorphism. The *rank* of (A, B, ϕ) is $|A| = |B|$. The composition of two morphisms is given by composition of functions, where the domain is the largest set on which the composition is defined, and the codomain is its bijective image. An *FI \sharp -module over k* is a functor from $\text{FI}\sharp$ to the category of k -modules.

Example 2.20. The most basic example is the $\text{FI}\sharp$ -module V taking a finite set S to the k -module V_S freely generated by the elements of S , with morphisms acting as follows. Given $i \in S$, let e_i be the corresponding basis element of V_S . Then the map $V_S \rightarrow V_T$ induced by a morphism $f = (A, B, \phi)$ from S to T is defined by $f(e_i) = e_{\phi(i)}$ if $i \in A$, and $f(e_i) = 0$ otherwise.

Note that if V is an $\text{FI}\sharp$ -module, restricting to the subset of morphisms in $\text{Hom}_{\text{FI}\sharp}(S, T)$ for which A is all of S gives V the structure of an FI -module, and restricting to those for which B is all of T gives V the structure of a co- FI -module. The relations in $\text{FI}\sharp$ impose conditions on how the FI -module and co- FI -module structures interact.

We point out that the endomorphisms of \mathbf{n} in $\text{FI}\sharp$ form the so-called *rook algebra* of rank n , and so V_n is a representation of the rook algebra. In particular, for the $\text{FI}\sharp$ -module V in Example 2.20, V_n is the standard representation of the rook algebra on k^n . The basic properties of the rook algebra and its representation theory were determined by Munn and Solomon [So].

The following elementary observation is one indication that $\text{FI}\sharp$ -modules are much more restrictive than mere FI -modules.

Proposition 2.21. *Let V be an $\text{FI}\sharp$ -module. Then for any injection $f: S \hookrightarrow T$, the induced map $f_*: V_S \rightarrow V_T$ is injective. In particular, whenever $m < n$, every map $f_*: V_m \rightarrow V_n$ is injective.*

Proof. Given $f \in \text{Hom}_{\text{FI}}(S, T) \subset \text{Hom}_{\text{FI}\sharp}(S, T)$, let $A = S$ and $B = f(A)$. Since f is injective, we can form the inverse map $f^{-1}: B \rightarrow A$ in $\text{Hom}_{\text{FI}\sharp}(T, S)$. Since $f^{-1} \circ f = \text{id}_S \in \text{Hom}_{\text{FI}\sharp}(S, S)$, the induced map f_* has a left inverse and thus is injective. \square

We say an $\text{FI}\sharp$ -module is *generated in degree $\leq m$* if every V_n is spanned by the images of maps of rank $\leq m$. Since every such image is the image of a map $V_m \rightarrow V_n$, this is equivalent to saying that V is generated in degree $\leq m$ as an FI -module. The following lemma will be used in our classification of $\text{FI}\sharp$ -modules.

Lemma 2.22. *Let V be an $\text{FI}\sharp$ -module. If $V_m = 0$ then all morphisms of $\text{FI}\sharp$ of rank $\leq m$ act as 0 on V .*

Proof. Let $(A, B, \phi) \in \text{Hom}_{\text{FI}\sharp}(S, T)$ be a morphism of rank $\leq m$. Then (A, B, ϕ) factors as a composition of a morphism in $\text{Hom}_{\text{FI}\sharp}(S, \mathbf{m})$ with a morphism in $\text{Hom}_{\text{FI}\sharp}(\mathbf{m}, T)$. This implies that the corresponding map $V_S \rightarrow V_T$ factors through V_m , which forces it to vanish. \square

In particular, this applies to the identity $\text{id}_{\mathbf{k}} \in \text{Hom}_{\text{FI}\sharp}(\mathbf{k}, \mathbf{k})$ for any $k < m$, and so we conclude that $V_k = 0$ for all $k < m$.

Example 2.23 ($M(W)$ is an $\text{FI}\sharp$ -module). The FI -module $M(m)$ carries a unique $\text{FI}\sharp$ -module structure extending the given FI -module structure. Recall that $M(m)_n$ is the free k -module spanned by $\text{Hom}_{\text{FI}}(\mathbf{m}, \mathbf{n})$. For each injection $f: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ let e_f be the corresponding basis element of $M(m)_n$, and write e for the element of $M(m)_m$ corresponding to the identity map. For notational simplicity we let $V := M(m)$, and we use the notation $f_*: V_n \rightarrow V_{n'}$ for the map induced by $f \in \text{Hom}_{\text{FI}\sharp}(\mathbf{n}, \mathbf{n}')$, so for example $e_f = f_*e$.

Given $g = (A, B, \phi) \in \text{Hom}_{\text{FI}\sharp}(\mathbf{n}, \mathbf{n}')$, we need to define the map $g_*: V_n \rightarrow V_{n'}$. It suffices to specify g_*e_f , and we make the following definition: if A contains the image of f , then $g \circ f$ is an injection $\{1, \dots, m\} \rightarrow \{1, \dots, n'\}$ and we define $g_*e_f = e_{g \circ f}$, while if A does not contain the image of f , we define $g_*e_f = 0$. Note that this is equivalent to saying that $g_*e_f = 0$ when $g \circ f$ in $\text{FI}\sharp$ has rank strictly less than m , and $g_*e_f = e_{g \circ f}$ when $g \circ f$ has rank m . From this description we see that $h_*(g_*(e_f)) = 0$ when $h \circ g \circ f$ has rank less than m and $h_*(g_*(e_f)) = e_{h \circ g \circ f}$ when $h \circ g \circ f$ has rank m .

Since this agrees with the definition of $(h \circ g)_* e_f$, our definition makes $M(m)$ into an $\text{FI}\sharp$ -module, as desired. We can now see that the $\text{FI}\sharp$ -module V of Example 2.20 is just $M(1)$.

Recall that precomposition with $\text{Hom}_{\text{FI}}(\mathbf{m}, \mathbf{m})$ gives an action of S_m on $M(m)$ as an FI -module, and this action evidently commutes with the $\text{FI}\sharp$ -module structure just defined. So if W is an S_m -module, then the FI -module

$$M(W) = M(m) \otimes_{k[S_m]} W$$

naturally carries the structure of an $\text{FI}\sharp$ -module as well. By abuse of notation we denote this $\text{FI}\sharp$ -module again by $M(W)$.

The extra rigidity imparted by an $\text{FI}\sharp$ -module structure makes the classification of $\text{FI}\sharp$ -modules substantially simpler than that of FI -modules.

Theorem 2.24 (Classification of $\text{FI}\sharp$ -modules). *Every $\text{FI}\sharp$ -module V is of the form*

$$V = \bigoplus_{i=0}^{\infty} M(W_i) \tag{11}$$

where W_i is a representation (possibly zero) of S_i .

Furthermore, if $V' = \bigoplus_i M(W'_i)$ is another $\text{FI}\sharp$ -module, any map $F: V \rightarrow V'$ of $\text{FI}\sharp$ -modules is of the form $\bigoplus_i M(F_i)$ for some maps $F_i: W_i \rightarrow W'_i$ of S_i -representations. That is, the functor

$$M(-): \bigoplus S_i\text{-Rep} \rightarrow \text{FI}\sharp\text{-Mod}$$

is an equivalence of categories, with inverse $H_0(-): \text{FI}\sharp\text{-Mod} \rightarrow \bigoplus S_i\text{-Rep}$.

Proof. We will prove the theorem by induction. Let V be an arbitrary $\text{FI}\sharp$ -module, and assume that $V_m = 0$ for all $m < n$. We define an endomorphism $E: V \rightarrow V$ of $\text{FI}\sharp$ -modules as follows. If S is a subset of T , we denote by I_S the morphism (S, S, id_S) in $\text{Hom}_{\text{FI}\sharp}(T, T)$. Then we define the action of E on V_T by

$$E_T = \sum_{\substack{S \subset T \\ |S|=n}} I_S$$

We first verify that $E: V \rightarrow V$ is a map of $\text{FI}\sharp$ -modules. Consider $f \in \text{Hom}_{\text{FI}\sharp}(T, R)$ defined by an isomorphism $T \supset A \xrightarrow{\phi} B \subset R$. By Lemma 2.22 our assumption implies that any morphism of rank less than n acts by 0, so we can compute that

$$f \circ E_T = \sum_{\substack{S \subset A \\ |S|=n}} \phi|_S \quad \text{and} \quad E_R \circ f = \sum_{\substack{S' \subset B \\ |S'|=n}} \phi|_{\phi^{-1}(S')}.$$

Since these sums coincide, we see that $f \circ E_T$ agrees with $E_R \circ f$ in its action on V_T , as desired.

Since E is a sum of morphisms of rank n , the image EV is contained in the sub- $\text{FI}\sharp$ -module of V generated by V_n . For any map $f \in \text{Hom}_{\text{FI}}(\mathbf{n}, T)$ we have $E \circ f = f$ by another application of Lemma 2.22, showing that E acts as the identity on the span of V_n . Combining these, we conclude that EV coincides with the sub- $\text{FI}\sharp$ -module of V generated by $(EV)_n = V_n$. Moreover, since E is idempotent it splits V as a direct sum $EV \oplus \ker[E]$. By Remark 2.13, there is a surjection $M(V_n) \twoheadrightarrow EV$ whose kernel we denote by K :

$$0 \rightarrow K \rightarrow M(V_n) \rightarrow EV \rightarrow 0$$

Since $M(V_n)$ is generated in degree n , the operator E acts as the identity on $M(V_n)$, whence also on K . This implies that $EK = K$. However EK is generated by K_n , which is zero since $M(V_n)_n \cong$

$(EV)_n \simeq V_n$, so we conclude that $K = 0$. We conclude that V splits as a direct sum $M(V_n) \oplus \ker[E]$, where $\ker[E]$ is zero in degrees below $n + 1$. The desired decomposition follows by induction on n .

If V' is another FI \sharp -module satisfying $V'_m = 0$ for $m < n$, we can define $E: V' \rightarrow V'$ in the same way as above. By definition, any map of FI \sharp -modules $F: V \rightarrow V'$ commutes with E and thus preserves the decompositions $V = M(V_n) \oplus \ker[E]$ and $V' = M(V'_n) \oplus \ker[E]$. By Proposition 2.6, any homomorphism $M(V_n) \rightarrow M(V'_n)$ is determined by a map $F_n: V_n \rightarrow V'_n$ of S_n -representations. The desired description of arbitrary maps of FI \sharp -modules follows by induction on n . Finally, the fact that H_0 is a left inverse for M —that is, that $H_0(M(W)) = W$ —is a general fact about FI-modules and was already noted in Definition 2.18. But the decomposition proved above shows that every FI \sharp -module V is of the form $V = M(W)$ for some W , and so we have $V = M(H_0(V))$ as well. \square

The classification in Theorem 2.24 has the following two corollaries.

Corollary 2.25. *Every sub-FI \sharp -module of $V = \bigoplus_i M(W_i)$ is of the form $V' = \bigoplus_i M(W'_i)$, where W'_i is a subrepresentation of W_i .*

Corollary 2.26. *If V is an FI \sharp -module generated in degree $\leq d$, then any sub-FI \sharp -module of V is also generated in degree $\leq d$.*

Theorem 2.24 also has the following consequence, which implies Theorem 1.8.

Corollary 2.27. *If V is an FI \sharp -module and k is a field of arbitrary characteristic,*

$$\begin{aligned} V \text{ is finitely generated} &\iff \dim_k V_n = O(n^d) \text{ for some } d \\ &\iff \dim_k V_n = P(n) \text{ for some polynomial } P \in \mathbb{Q}[T] \text{ and all } n \geq 0 \end{aligned}$$

For k an arbitrary commutative ring, we still have

$$V \text{ is finitely generated} \iff V_n \text{ is generated by } O(n^d) \text{ elements for some } d$$

Proof. If W_i is a finite-dimensional representation of S_i over a field k , we saw in (8) that

$$\dim M(W_i)_n = \dim W_i \cdot \binom{n}{i} = O(n^i).$$

Thus the sequence of dimensions $\dim V_n$ is bounded by a polynomial if and only if the sum in (11) is finite and each W_i is finite-dimensional, i.e. if V is finitely generated. In this case we have $\dim V_n = \sum_i \dim W_i \cdot \binom{n}{i} = P(n)$ for all $n \geq 0$, as desired. Over an arbitrary ring k the situation is not much harder. We still have the decomposition $M(W_i)_n \simeq W_i^{\oplus \binom{n}{i}}$ as k -modules, which shows that for a finitely generated FI \sharp -module V the k -module V_n is generated by $O(n^d)$ elements. For the converse, suppose that V_n is generated by $O(n^d)$ elements, so V_n admits a surjection from k^{cn^d} for some constant c . Suppose furthermore that W_i is nonzero for some $i > 0$, and let \mathfrak{m} be a maximal ideal of k such that $W_i/\mathfrak{m}W_i \neq 0$. Then the hypothesized surjection shows that the (k/\mathfrak{m}) -dimension of k^{cn^d}/\mathfrak{m} is at least that of $W_i^{\oplus \binom{n}{i}}/\mathfrak{m}$ for all n , which immediately implies $i \leq d$. So W_i is zero for all $i > d$, and is finitely generated for all $i \leq d$ by hypothesis, so V is finitely generated. \square

Tensor products of FI \sharp -modules. Let k be a field of characteristic 0, and recall that $M(\lambda) = M(V_\lambda)$ is the free FI \sharp -module generated by the single irreducible representation V_λ . Since the tensor product of two FI \sharp -modules is an FI \sharp -module, Theorem 2.24 implies that every tensor product $M(\lambda) \otimes M(\mu)$

decomposes as a direct sum of finitely many FI \sharp -modules $M(\nu)$. It is not hard to compute small examples by hand. For instance, we have:

$$\begin{aligned}
M(\square) \otimes M(\square) &= M(\square\square) \oplus M(\begin{smallmatrix} \square \\ \square \end{smallmatrix}) \oplus M(\square) \\
M(\square) \otimes M(\square\square) &= M(\square\square\square) \oplus M(\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}) \oplus M(\square\square) \oplus M(\begin{smallmatrix} \square \\ \square \end{smallmatrix}) \\
M(\square\square) \otimes M(\square\square) &= M(\square\square\square\square) \oplus M(\begin{smallmatrix} \square & \square & \square \\ \square \end{smallmatrix}) \oplus M(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}) \\
&\quad \oplus M(\square\square\square) \oplus M(\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix})^{\oplus 2} \oplus M(\begin{smallmatrix} \square \\ \square & \square \end{smallmatrix}) \oplus M(\square\square).
\end{aligned} \tag{12}$$

These computations have natural combinatorial interpretations. For example, $M(\square) \otimes M(\square)$ takes a finite set S to the vector space spanned by ordered pairs (x, y) taken from S . This naturally splits into those pairs with $x = y$, yielding the summand $M(\square)$, and those with $x \neq y$, yielding the summand $M(2) = M(\square\square) \oplus M(\begin{smallmatrix} \square \\ \square \end{smallmatrix})$. For any partitions λ and μ we have a direct sum decomposition

$$M(\lambda) \otimes M(\mu) = \bigoplus_{\nu} d_{\lambda, \mu}^{\nu} M(\nu) \tag{13}$$

The coefficients $d_{\lambda, \mu}^{\nu}$ are nonnegative integers, and it can be shown that $d_{\lambda, \mu}^{\nu}$ is only nonzero when $\max(|\lambda|, |\mu|) \leq |\nu| \leq |\lambda| + |\mu|$. It is straightforward to check that when $|\lambda| = |\mu| = |\nu| = n$, the coefficient $d_{\lambda, \mu}^{\nu}$ is equal to the Kronecker coefficient (the multiplicity of V_{ν} in $V_{\lambda} \otimes V_{\mu}$).

Moreover, any Schur functor \mathbb{S}_{λ} yields an FI \sharp -module $\mathbb{S}_{\lambda}(M(\square))$ whose “leading term” is $M(\lambda)$, in the sense that $\mathbb{S}_{\lambda}(M(\square)) = M(\lambda) \oplus V$ where V is generated in degrees $< |\lambda|$. It follows that when $|\nu| = |\lambda| + |\mu|$, the coefficient $d_{\lambda, \mu}^{\nu}$ is equal to the Littlewood–Richardson coefficient $c_{\lambda, \mu}^{\nu}$ (the coefficient of $\mathbb{S}_{\nu}W$ in $\mathbb{S}_{\lambda}W \otimes \mathbb{S}_{\mu}W$). The honeycomb model used by Knutson–Tao in their proof of the saturation conjecture [KT] gives a geometric interpretation for the Littlewood–Richardson coefficients $c_{\lambda, \mu}^{\nu}$ as the number of integer points in a certain Berenstein–Zelevinsky polytope. It would be very interesting to find a similar geometric interpretation for the coefficients $d_{\lambda, \mu}^{\nu}$.

Problem 2.28 (FI \sharp -module tensor coefficients). *Give a geometric interpretation of the structural coefficients $d_{\lambda, \mu}^{\nu}$ in (13). Give a method for determining which of these coefficients are nonzero.*

Projective FI-modules. A *projective FI-module* is a projective object in the category FI-Mod. It follows from general considerations (see [Wei, Exercise 2.3.8]) that the category FI-Mod has enough projectives. In the companion paper [CEF], we investigate the *higher homology* $H_i(V)$ of an FI-module V . The functors H_i are defined as the derived functors of H_0 , and can be computed from a projective resolution of V . For example, for the FI-module V from Remark 2.10, we can compute from the projective resolution in (10) that $H_i(V)$ is the sign representation ε_i of S_i .

Any FI-module of the form $M(P_n)$ where P_n is a projective $k[S_n]$ -module is projective (since $M(-)$ is the left adjoint to an exact functor and thus preserves projectives [Wei, Proposition 2.3.10]). We prove in [CEF] that these are exactly the projective FI-modules; this theorem is not used in the present paper (except in Remark 2.68, which can be ignored), so we do not include the proof here.

Theorem 2.29 ([CEF]). *The projective FI-modules are precisely the sums $\bigoplus_{n \geq 0} M(P_n)$, where P_n is a projective $k[S_n]$ -module.*

In particular, when k is a field of characteristic 0, Theorem 2.29 implies that the projective FI-modules are precisely those that can be extended to FI \sharp -modules. As a consequence, the tensor product of two projective FI-modules over a field of characteristic 0 is projective.

2.4 Stability degree

In this section we introduce the notion of a *stability degree* for an FI-module. It should be thought of as a counterpart to the “stable range” for representation stability in [CF]; see Proposition 2.58 for a precise comparison.

Shifts on FI-modules. Given an FI-module V and a natural number $a \geq 1$, we define the “shifted” FI-module $S_{+a}V$ in the following way. There is a functor $\coprod_{[-a]}: \text{FI} \rightarrow \text{FI}$ which takes a finite set T to the disjoint union $\{-1, \dots, -a\} \coprod T$, and takes an inclusion $f: T \hookrightarrow T'$ to the induced inclusion $\{-1, \dots, -a\} \coprod T \hookrightarrow \{-1, \dots, -a\} \coprod T'$. (The specific set $\{-1, \dots, -a\}$ is irrelevant; this one is chosen to minimize mental collisions with the finite sets T the reader is most likely to have in mind.)

Definition 2.30 (Shifted FI-modules). Given any FI-module $V: \text{FI} \rightarrow k\text{-Mod}$, we define the FI-module $S_{+a}V: \text{FI} \rightarrow k\text{-Mod}$ to be the composition

$$S_{+a}V := V \circ \coprod_{[-a]}.$$

In particular, $(S_{+a}V)_n$ is isomorphic to V_{n+a} as a representation of S_n .

The functor $S_{+a}: \text{FI-Mod} \rightarrow \text{FI-Mod}$ is exact, since kernels and cokernels in FI-Mod are computed pointwise. The action of the symmetric group S_a on $\{-1, \dots, -a\}$ induces an action of S_a on the FI-module $S_{+a}V$.

Proposition 2.31. *If V is generated in degree $\leq d$, then $S_{+a}V$ is generated in degree $\leq d$. If V is finitely generated, then $S_{+a}V$ is finitely generated.*

We remark that the original generating set for V does not suffice to generate $S_{+a}V$, so Proposition 2.31 is not obvious; we will need to expand the size of our generating set by a factor which, although finite, grows with a .

Proof. By Remark 2.13, any FI-module V has a surjection

$$\bigoplus_{m \geq 0} M(m)^{\oplus V_m} \twoheadrightarrow V,$$

and this induces a surjection

$$\bigoplus_{m \geq 0} S_{+a}M(m)^{\oplus V_m} \twoheadrightarrow S_{+a}V.$$

If V is generated in degree $\leq d$, we can replace this direct sum by a sum over $0 \leq m \leq d$, and if V is finitely generated we can replace it by a finite direct sum. It thus suffices to prove that $S_{+a}M(m)$ is generated by $S_{+a}M(m)_m$, since this is a free k -module of finite rank. Specifically, $S_{+a}M(m)_n$ is the free k -module spanned by the injections $\{1, \dots, m\} \rightarrow \{-a, \dots, -1, 1, \dots, n\}$. When $n \geq m$, any such injection is the composition of an injection $\{1, \dots, m\} \rightarrow \{-a, \dots, -1, 1, \dots, m\}$ with an injection $\{-a, \dots, -1, 1, \dots, m\} \rightarrow \{-a, \dots, -1, 1, \dots, n\}$ that restricts to the identity on $\{-a, \dots, -1\}$. Thus every element of $S_{+a}M(m)_n$ is in the span of $S_{+a}M(m)_m$, as desired. \square

Coinvariants of FI-modules. If V_n is a representation of S_n , the coinvariant quotient $(V_n)_{S_n}$ is the k -module $V_n \otimes_{k[S_n]} k$; this is the largest S_n -equivariant quotient of V_n on which S_n acts trivially. A central tool in our analysis of FI-modules is to perform these quotients simultaneously for all n .

Definition 2.32 (The FI-module τV). Given an FI-module V , let τV be the FI-module defined by $(\tau V)_n := (V_n)_{S_n}$, where the map $f_*: (\tau V)_m \rightarrow (\tau V)_n$ is the map $(V_m)_{S_m} \rightarrow (V_n)_{S_n}$ induced by $f_*: V_m \rightarrow V_n$. There is a natural surjection of FI-modules $V \twoheadrightarrow \tau V$.

Any two inclusions $f, f': \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ are equivalent under post-composition with S_n , so the induced maps $f_*, f'_*: (V_m)_{S_m} \rightarrow (V_n)_{S_n}$ are necessarily the same. The functor taking an S_n -representation V_n to the coinvariants $(V_n)_{S_n}$, which is automatically right exact, is also left exact when $|S_n| = n!$ is invertible in k (by the transfer map). It follows that $\tau: \text{FI-Mod} \rightarrow \text{FI-Mod}$ is an exact functor when k is a field of characteristic 0, or more generally any ring containing \mathbb{Q} . We point out that when FI-modules are thought of as modules for the exponential twisted commutative algebra as in [AM], the functor τ is precisely the *bosonic Fock functor* that plays a crucial role in that theory.

Stability degree.

Definition 2.33 (The graded $k[T]$ -module $\Phi_a(V)$). Fix an integer $a \geq 0$, and let V be an FI-module. We define $\Phi_a(V)$ to be the graded $k[T]$ -module whose underlying k -vector space is the direct sum of the coinvariants

$$\Phi_a(V) = \bigoplus_{n \geq 0} \Phi_a(V)_n := \bigoplus_{n \geq 0} (V_{n+a})_{S_n},$$

and on which $T: \Phi_a(V)_n \rightarrow \Phi_a(V)_{n+1}$ acts by the map $(V_{n+a})_{S_n} \rightarrow (V_{n+1+a})_{S_{n+1}}$ induced by $f_*: V_{n+a} \rightarrow V_{n+1+a}$ for some (equivalently, any) $f \in \text{Hom}_{\text{FI}}(\mathbf{n} + \mathbf{a}, \mathbf{n} + 1 + \mathbf{a})$.

Recall that the shifted FI-module $S_{+a}V$ has $(S_{+a}V)_n = V_{n+a}$, so the functor $\Phi_a: \text{FI-Mod} \rightarrow k[T]\text{-Mod}$ is essentially the composition $\tau \circ S_{+a}$ (except that technically the latter has values in FI-Mod). When k contains \mathbb{Q} , the functor Φ_a is exact (because τ and S_{+a} are both exact). The natural action of S_a on $S_{+a}V$ induces an action of S_a on $\tau S_{+a}V$, and thus on $\Phi_a(V)$.

Definition 2.34 (Stability degree). We say that an FI-module V has *stability degree* $\leq s$ if for all $a \geq 0$, the map $\Phi_a(V)_n \rightarrow \Phi_a(V)_{n+1}$ induced by multiplication by T is an isomorphism (of k -modules, or equivalently of S_a -representations) for all $n \geq s$.

Equivalently, an FI-module V has stability degree $\leq s$ if for all $a \geq 0$ the maps $(V_n)_{S_{n-a}} \rightarrow (V_{n+1})_{S_{n+1-a}}$ induced by $I_n: \{1, \dots, n\} \hookrightarrow \{1, \dots, n+1\}$ are isomorphisms for all $n \geq s+a$.

It turns out to be useful to refine the notion of stability degree slightly. This will be especially important in Section 4 when we study the behavior of FI-modules in spectral sequences.

Definition 2.35 (Injectivity degree and surjectivity degree). We say that V has *injectivity degree* $\leq s$ (resp. *surjectivity degree* $\leq s$) if for all $a \geq 0$, the map $\Phi_a(V)_n \rightarrow \Phi_a(V)_{n+1}$ induced by multiplication by T is injective (resp. surjective) for all $n \geq s$.

By definition, the maximum of an injectivity degree and a surjectivity degree for V is a stability degree for V .

Remark 2.36. Given a surjection of FI-modules $V \twoheadrightarrow W$, if V has surjectivity degree $\leq s$ then W has surjectivity degree $\leq s$ as well. This uses the fact that Φ_a is right-exact over arbitrary rings. Since Φ_a is exact when k contains \mathbb{Q} , we have the following. If k contains \mathbb{Q} and V is an FI-module with injectivity degree $\leq s$, then any sub-FI-module W of V has injectivity degree $\leq s$ as well.

Before moving on, we give an example where the injectivity degree and surjectivity degree differ quite drastically.

Proposition 2.37. *The FI-module $M(m)$ has injectivity degree 0 and surjectivity degree m .*

Proof. The FI-module $M(m)$ carries an S_m -action which descends to an action on $\Phi_a M(m)$. Recall that $S_{+a}M(m)_n$ is spanned freely by the injections

$$\{1, \dots, m\} \rightarrow \{-a, \dots, -1, 1, \dots, n\}.$$

So $(\Phi_a M(m))_n$, by definition the S_n -coinvariants of $S_{+a}M(m)_n$, is spanned freely by the set of

$$\{1, \dots, m\} \rightarrow \{-a, \dots, -1, \star\}$$

injective except at \star and sending at most n elements of $\{1, \dots, m\}$ to \star . Denote this set by $B_{\leq n}$. The natural inclusion from $B_{\leq n}$ to $B_{\leq n+1}$ is injective for all n . Moreover when $n \geq m$ the condition on the preimage of \star is vacuous, and so $B_{\leq n} = B_{\leq n+1} = B_{\leq m}$ for all $n \geq m$. \square

Since $(\Phi_0 M(m))_m = (M(m)_m)_{S_m} \simeq k$ while $M(m)_{m-1} = 0$, the bound on surjectivity degree in Proposition 2.37 is sharp. We can extend this result to any FI-module of the form $M(W)$.

Proposition 2.38. *For any $k[S_m]$ -module W , the FI-module $M(W)$ has injectivity degree 0 and surjectivity degree $\leq m$.*

Proof. Recall that $M(W) = W \otimes_{k[S_m]} M(m)$. The action of S_m on $M(m)$ commutes with the shift functor S_{+a} and with the action of $S_n = \text{Hom}_{\text{FI}}(\mathbf{n}, \mathbf{n})$, so we have $S_{+a}M(W) = W \otimes_{k[S_m]} S_{+a}M(m)$ and $\Phi_a M(W) = W \otimes_{k[S_m]} \Phi_a M(m)$. Maintaining the notation of the proof of Proposition 2.37, we saw above that $\Phi_a M(m)_n$ is isomorphic as an S_m -representation to $k[B_{\leq n}]$. Since $B_{\leq n}$ is an S_m -invariant subset of $B_{\leq n+1}$, the map of S_m -representations $\Phi_a M(m)_n \rightarrow \Phi_a M(m)_{n+1}$ embeds the former as a direct summand of the latter for all $n \geq 0$. Since $B_{\leq n} = B_{\leq n+1}$ for $n \geq m$, the map

$$\Phi_a M(W)_n = W \otimes_{k[S_m]} \Phi_a M(m)_n \longrightarrow W \otimes_{k[S_m]} \Phi_a M(m)_{n+1} = \Phi_a M(W)_{n+1}$$

is injective for all $n \geq 0$ and an isomorphism for all $n \geq m$. \square

Proposition 2.39. *If V is generated in degree $\leq d$ then V has surjectivity degree $\leq d$.*

Proof. If V is generated in degree $\leq d$, Remark 2.13 shows that V is a quotient of $\bigoplus_{m \leq d} M(V_m)$. The latter FI-module has surjectivity degree $\leq d$ by Proposition 2.37, so V has surjectivity degree $\leq d$ by Remark 2.36. \square

2.5 Stability degree in characteristic 0

In this section we investigate some structural properties of FI-modules over k which hold when k is a field of characteristic 0, or in some cases any ring containing \mathbb{Q} . We begin by recording some elementary observations relating the structure of S_n -representations to the structure of their S_{n-a} -coinvariants.

Lemma 2.40. *Given an irreducible S_n -representation $V(\lambda)_n$ and some $a \leq n$, consider the S_{n-a} -coinvariants $(V(\lambda)_n)_{S_{n-a}}$ as a representation of S_a .*

- (i) *We have $(V(\lambda)_n)_{S_{n-a}} = 0 \iff a < |\lambda|$.*
- (ii) *If $a = |\lambda|$, we have $(V(\lambda)_n)_{S_{n-a}} \simeq V_\lambda$ for all $n \geq |\lambda| + \lambda_1$.*
- (iii) *In general, for fixed a the S_a -representation $(V(\lambda)_n)_{S_{n-a}}$ is independent of n once $n \geq a + \lambda_1$.*
- (iv) *If every irreducible subrepresentation $V(\lambda)_n$ of an S_n -representation V_n satisfies $|\lambda| \leq a$, then $V_n = 0 \iff (V_n)_{S_{n-a}} = 0$.*

Proof. The partition $\lambda[n]$ is obtained from λ by adding one box to each of the first $n - |\lambda|$ columns. The branching rule states that $(V_{\lambda[n]})_{S_{n-a}}$ is the sum of V_μ over partitions $\mu \vdash a$ which can be obtained from $\lambda[n]$ by removing $n - a$ boxes, at most one from each column. When $a < |\lambda|$ this is impossible, since $\lambda[n]$ has only $n - |\lambda|$ columns, and when $a = |\lambda|$ we obtain just λ again; this demonstrates (i) and (ii). As long as $n - a \geq \lambda_1$, the μ which occur are exactly those $\mu \vdash a$ obtained from λ by adding at most one box to each column, which demonstrates (iii). Finally, (iv) follows immediately from (i). \square

When k is a field of characteristic 0, Proposition 2.38 implies that the stability degree of the FI-module $M(\lambda) = M(V_\lambda)$ is $\leq |\lambda|$, but in fact this can usually be improved. Recall that λ_1 is the length of the first row of the partition λ .

Proposition 2.41. *Let k be a field of characteristic 0. For any partition λ the FI-module $M(\lambda)$ has stability degree $\leq \lambda_1$.*

Proof. By Proposition 2.38, $M(\lambda)$ has injectivity degree 0. This means that the map $T: \Phi_a(M(\lambda))_n \rightarrow \Phi_a(M(\lambda))_{n+1}$ is injective for all $n \geq 0$. Thus to show that this map is an isomorphism for all $n \geq \lambda_1$, it suffices to show that $\dim_k \Phi_a(M(\lambda))_n$ is constant for $n \geq \lambda_1$.

By definition, $\Phi_a(M(\lambda))_n$ is the S_a -representation $(M(\lambda)_{n+a})_{S_n}$. By the branching rule $M(\lambda)_{n+a}$ is the sum of V_μ over all partitions $\mu \vdash n+a$ obtained from λ by adding $n+a - |\lambda|$ boxes, no two in the same column. Similarly, the branching rule states that $(V_\mu)_{S_n}$ is the sum of V_ν over all partitions $\nu \vdash a$ obtained from μ by removing n boxes, no two in the same column. Say that μ is *valid* for ν if these conditions are satisfied. Then the multiplicity of V_ν in $\Phi_a(M(\lambda))_n$ is the number of partitions $\mu \vdash n+a$ that are valid for ν .

Given $\mu \vdash n+a$, define $\mu' \vdash n+a+1$ by $\mu' = (\mu_1 + 1, \mu_2, \dots, \mu_\ell)$. It is easy to check that if μ is valid for ν , then μ' is also valid for ν . Conversely, let $n \geq \lambda_1$, and assume that $\eta \vdash n+a+1$ is valid for ν . Since ν is obtained from η by removing $n+1$ boxes, no two in the same column, η must have at least $n+1$ columns. Since $n+1 > \lambda_1$, at least one box was added to the first row of λ to produce η . Since we cannot add another box to the same column, we conclude that $\eta_1 > \eta_2$. This implies that η can be written as $\eta = \mu'$ for some $\mu \vdash n+a$ that is valid for ν . Thus the multiplicity of V_ν in $\Phi_a(M(\lambda))_n$ and in $\Phi_a(M(\lambda))_{n+1}$ is the same once $n \geq \lambda_1$.

We conclude that $\Phi_a(M(\lambda))_n$ is isomorphic as an S_a -representation to $\Phi_a(M(\lambda))_{n+1}$ when $n \geq \lambda_1$, and since multiplication by T is always injective, it is an isomorphism when $n \geq \lambda_1$. \square

A bound on the stability degree imposes a condition on the width of the irreducible constituents of the representations V_n .

Proposition 2.42. *If V is an FI-module over a field of characteristic 0 with stability degree $\leq s$, then for every $n \geq 0$ and every irreducible constituent $V(\lambda)_n$ of the S_n -representation V_n , we have $\lambda_1 \leq s$.*

Before proving Proposition 2.42, we first need to establish the following.

Proposition 2.43. *When k is a field of characteristic 0, there is a (non-canonical) surjection:*

$$M(H_0(V)) \twoheadrightarrow V \tag{14}$$

Proof. We construct this surjection inductively. Recall that a map from $M(H_0(V))_n$ to V is determined by a map from $H_0(V)_n$ to V_n . In particular, since $H_0(V)_0 = V_0$, the first map $M(H_0(V))_0 \rightarrow V$ is determined, and is a surjection onto V_0 . Assume by induction that we have defined a map $\bigoplus_{k < n} M(H_0(V))_k \rightarrow V$ and it is a surjection in degree k for all $k < n$. By definition, the image of this map in degree n will be $\text{span}(V_{<n})_n \subset V_n$. Since k has characteristic 0, the short exact sequence of $k[S_n]$ -modules

$$0 \rightarrow \text{span}(V_{<n})_n \rightarrow V_n \rightarrow H_0(V)_n \rightarrow 0$$

admits a section $H_0(V)_n \rightarrow V_n$, and we take the corresponding map $M(H_0(V))_n \rightarrow V$. By construction, the resulting map $\bigoplus_{k \leq n} M(H_0(V))_k \rightarrow V$ is a surjection in degree k for all $k \leq n$. By induction on n , this completes the proof. \square

Proof of Proposition 2.42. We first show that every irreducible constituent $V(\lambda)_n$ of $M(V_\mu)_n$ satisfies $\lambda_1 \leq \mu_1$. We saw in the proof of Proposition 2.41 that the V_ν occurring in $M(\mu)_n = \text{Ind}_{S_m \times S_{n-m}}^{S_n} V_\mu \boxtimes k$ are exactly those ν obtained from μ by adding $n - m$ boxes, no two in the same column. In particular, the length of the *second* row of ν is bounded by μ_1 , the length of the first row of μ . When we write $V_\nu = V(\lambda)_n$ we have $\lambda_1 = \nu_2$, which verifies the claim.

We now prove that if V has stability degree $\leq s$, every irreducible constituent V_μ of $H_0(V)_m$ satisfies $\mu_1 \leq s$. By the previous paragraph, this will imply that every irreducible constituent $V(\lambda)_n$ of $M(H_0(V))_n$ satisfies $\lambda_1 \leq s$ (note the change of indexing). By Proposition 2.43, V is a quotient of $M(H_0(V))$, so we conclude that every irreducible constituent $V(\lambda)_n$ of V_n satisfies $\lambda_1 \leq s$, as desired.

Let $H_0(V)^{\text{FI}}$ denote $H_0(V)$ considered as an FI-module, where every morphism in $\text{Hom}_{\text{FI}}(\mathbf{m}, \mathbf{n})$ with $m \neq n$ acts by 0. The FI-module $H_0(V)^{\text{FI}}$ can be characterized as the maximal quotient of V for which all maps $H_0(V)_n^{\text{FI}} \rightarrow H_0(V)_{n+1}^{\text{FI}}$ are zero. There is a canonical surjection of FI-modules $V \twoheadrightarrow H_0(V)^{\text{FI}}$. Since Φ_a is right exact, we have a surjection $\Phi_a(V) \twoheadrightarrow \Phi_a(H_0(V)^{\text{FI}})$. Thus by Remark 2.36, the surjectivity degree of $H_0(V)^{\text{FI}}$ is $\leq s$.

If $H_0(V)_m$ contains V_μ , write $V_\mu = V(\nu)_m$ and let $a = m - \mu_1 = |\nu|$. By definition, $\Phi_a(H_0(V)^{\text{FI}})_{\mu_1} = (H_0(V)_m)_{S_{m-a}}$. Since Φ_a is exact in characteristic 0 this contains $(V(\nu)_m)_{S_{m-a}}$, which by Lemma 2.40(ii) is $V_\nu \neq 0$. This shows that $\Phi_a(H_0(V)^{\text{FI}})_{\mu_1} \neq 0$. However, T acts by 0 on $\Phi_a(H_0(V)^{\text{FI}})$, so $T: \Phi_a(H_0(V)^{\text{FI}})_{\mu_1-1} \rightarrow \Phi_a(H_0(V)^{\text{FI}})_{\mu_1}$ is definitely not surjective. Thus the surjectivity degree of $H_0(V)^{\text{FI}}$, which we proved above is at most s , is at least μ_1 . This proves that $\mu_1 \leq s$, as desired. \square

Homological properties of stability degree. We now give three easy technical propositions which govern the behavior of stability degree under various algebraic constructions. In all three, we assume that k contains \mathbb{Q} so that the functor Φ_a is exact.

Proposition 2.44. *Let $f: V \rightarrow W$ be a morphism of FI-modules and assume that k contains \mathbb{Q} . Suppose that V has injectivity degree $\leq B$ and surjectivity degree $\leq C$, and W has injectivity degree $\leq D$ and surjectivity degree $\leq E$. Then $\ker f$ has injectivity degree $\leq B$ and surjectivity degree $\leq \max(C, D)$, and $\text{coker } f$ has injectivity degree $\leq \max(C, D)$ and surjectivity degree $\leq E$.*

Proof. Consider the following diagram, where the vertical maps are multiplication by T and the rows are exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Phi_a(\ker f) & \longrightarrow & \Phi_a V & \longrightarrow & \Phi_a W \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Phi_a(\ker f) & \longrightarrow & \Phi_a V & \longrightarrow & \Phi_a W \end{array}$$

It is clear the leftmost vertical map is injective whenever the middle vertical map is injective. A simple diagram chase shows that the leftmost vertical map is surjective whenever the middle vertical map is surjective and the rightmost vertical map is injective. This yields the first assertion; the second is proved in the same way, applying Φ_a to the exact sequence $V \rightarrow W \rightarrow \text{coker } f \rightarrow 0$. \square

We can apply this proposition to a complex of FI-modules, meaning a sequence $U \rightarrow V \rightarrow W$ where the composition $U \rightarrow W$ is zero.

Proposition 2.45. *Let $U \xrightarrow{f} V \xrightarrow{g} W$ be a complex of FI-modules, and assume that k contains \mathbb{Q} . Assume that U has surjectivity degree $\leq A$, that V has injectivity degree $\leq B$ and surjectivity degree $\leq C$, and that W has injectivity degree $\leq D$. Then the FI-module $\ker g / \text{im } f$ has injectivity degree $\leq \max(A, B)$ and surjectivity degree $\leq \max(C, D)$.*

Proof. By Proposition 2.44 we know that $\ker g$ has injectivity degree $\leq B$ and surjectivity degree $\leq \max(C, D)$. The FI-module $\ker g / \operatorname{im} f$ is the cokernel of the map $U \rightarrow \ker g$, so another application of Proposition 2.44 yields the desired result. \square

Proposition 2.46. *Let V be an FI-module with a filtration $V = F_0V \supset F_1V \supset \cdots \supset F_jV = 0$ by FI-modules F_iV , and assume that k contains \mathbb{Q} . If $F_iV/F_{i+1}V$ has injectivity degree $\leq A$ and surjectivity degree $\leq B$ for all i , then V has injectivity degree $\leq A$ and surjectivity degree $\leq B$.*

Proof. By induction on j it suffices to consider the case where $j = 2$, so that V sits in an exact sequence

$$0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0.$$

Applying the “four lemma” (or an elementary diagram chase) to

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Phi_a U & \longrightarrow & \Phi_a V & \longrightarrow & \Phi_a W \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Phi_a U & \longrightarrow & \Phi_a V & \longrightarrow & \Phi_a W \longrightarrow 0 \end{array}$$

shows that the middle vertical arrow is injective (resp. surjective) whenever the rightmost and leftmost vertical arrows are. \square

Proposition 2.47. *Assume that k contains \mathbb{Q} . Let W be a sub-FI-module of an FI-module V . If V has injectivity degree $\leq N$ and W is generated in degree $\leq d$, then W has stability degree $\leq \max(N, d)$.*

Proof. W has surjectivity degree $\leq d$ by Proposition 2.39, and has injectivity degree $\leq N$ by Remark 2.36. \square

Finite presentations and stability degree.

Definition 2.48 (Finitely presented FI-module). We say an FI-module V is *presented* with *generator degree* $\leq g$ and *relation degree* $\leq r$ if there exists a surjection

$$\bigoplus_{i=0}^g M(W_i) \twoheadrightarrow V \tag{15}$$

whose kernel K is generated in degree $\leq r$; we say that V is presented in finite degree if this holds for some g and r . We say that V is *finitely presented* with generator degree $\leq g$ and relation degree $\leq r$ if we have a surjection (15) where each W_i is a finitely generated k -module and K is finitely generated in degree $\leq r$.

The Noetherian property, which we will prove later in Theorem 2.60, implies that every finitely generated FI-module over a field of characteristic 0 is finitely presented in this sense. But we can already show that a FI-module presented in finite degree admits an explicit stability degree.

Theorem 2.49 (Stability degree for f.p. FI-modules). *Let V be an FI-module over a ring containing \mathbb{Q} , and suppose V is presented with generator degree $\leq g$ and relation degree $\leq r$. Then V has stability degree $\leq \max(r, g)$.*

Proof. By Proposition 2.38, $M(\bigoplus W_i)$ has injectivity degree 0 and surjectivity degree $\leq g$. Since K is generated in degree $\leq r$, Proposition 2.47 implies that K has stability degree $\leq r$. Finally, Proposition 2.44 implies that the quotient V has stability degree $\leq \max(g, r)$, as desired. \square

We note that if k is a field of characteristic 0, and V is an FI-module such that $H_0(V)$ vanishes in degree above g and $H_1(V)$ (defined in §2.3) vanishes in degree above r , then V is presented with generator degree $\leq g$ and relation degree $\leq r$.

The weight of an FI-module.

Definition 2.50 (Weight of an FI-module). Let V be an FI-module over a field of characteristic 0. We say that V has *weight* $\leq d$ if for every $n \geq 0$ and every irreducible constituent $V(\lambda)_n$ of V_n , we have $|\lambda| \leq d$.

We denote by $\text{weight}(V)$ the smallest nonnegative integer d for which V is of weight $\leq d$, or ∞ if there is no such d . Note that if V has weight $\leq d$, the same is true for any subquotient of V .

Proposition 2.51. *Let V be an FI-module over a field of characteristic 0. If V is generated in degree $\leq d$, then V has weight $\leq d$.*

Proof. By Remark 2.13, the hypothesis implies that V is a quotient of $\bigoplus_{m \leq d} M(m)^{\oplus V_m}$. Since the conclusion is preserved under direct sum and quotients, it suffices to prove that $M(m)$ has weight $\leq d$ for $m \leq d$. Without loss of generality, it suffices to prove that $M(d)$ has weight $\leq d$.

By (7) we have $M(d)_n \simeq \text{Ind}_{S_{n-d}}^{S_n} k$, which by Pieri's rule [FH, Exercise 4.44] is the sum of $V(\lambda)_n$ over λ for which $\lambda[n]$ is obtained from the trivial partition $(n-d)$ by adding d boxes. In particular, the first term of $\lambda[n]$, which by definition is $n - |\lambda|$, must be at least $n - d$. We conclude that $|\lambda| \leq d$ for any irreducible constituent $V(\lambda)_n$ of $M(d)_n$, as desired. \square

The weight of an FI-module V yields a criterion to detect when $\Phi_a(V)$ vanishes.

Proposition 2.52. *Let V be an FI-module over a field of characteristic 0 of weight $\leq a$. Then $\Phi_a(V)$ is a trivial $k[T]$ -module if and only if V_n vanishes for all $n \geq a$.*

Proof. The assumption that $\text{weight}(V) \leq a$ guarantees that Lemma 2.40(iv) applies, so for all $n \geq a$ we have $\Phi_a(V)_{n-a} = (V_n)_{S_{n-a}} = 0 \iff V_n = 0$. \square

There is no way to bound the stability degree of an FI-module in terms of its weight, but for FI \sharp -modules we have the following sharp bound.

Proposition 2.53. *Let V be an FI \sharp -module over a field of characteristic 0. If V has weight $\leq d$, then V has stability degree $\leq d$.*

Proof. Write V as a direct sum $\bigoplus_{\lambda} M(\lambda)^{\oplus c_{\lambda}}$. Since $M(\lambda)_n$ contains $V(\lambda)_n$ as an irreducible constituent for sufficiently large n (in fact, for all $n \geq |\lambda| + \lambda_1$) we see that $|\lambda| \leq d$ for every $M(\lambda)$ appearing in the direct sum decomposition of V . By Proposition 2.41, the stability degree of $M(\lambda)$ is $\lambda_1 \leq |\lambda| \leq d$. \square

Proposition 2.54. *Fix $d \geq 0$, and let V be an FI-module over a field of characteristic 0. There is a sub-FI-module $\tau_{\geq d}V$ of V such that $(\tau_{\geq d}V)_n$ consists of all $V(\lambda)_n$ -isotypic components of V_n for which $|\lambda| \geq d$.*

Proof. This description characterizes $(\tau_{\geq d}V)_n$ uniquely, so we need only show that these subspaces define a sub-FI-module $\tau_{\geq d}V$. That is we must show that for every $f \in \text{Hom}_{\text{FI}}(\mathbf{m}, \mathbf{n})$:

$$f_*((\tau_{\geq d}V)_m) \subset (\tau_{\geq d}V)_n \quad (16)$$

Moreover, since $(\tau_{\geq d}V)_m$ and $(\tau_{\geq d}V)_n$ are S_m -invariant and S_n -invariant, respectively, it suffices to demonstrate (16) for $f = I_{m,n}$. Assume otherwise; then there is some irreducible constituent $V(\lambda)_m$ of V_m with $|\lambda| \geq d$ for which the S_m -equivariant map $I_{m,n}: V(\lambda)_m \rightarrow V_n/(\tau_{\geq d}V)_n$ is nonzero.

Every S_n -irreducible constituent $V(\mu)_n$ of $V_n/(\tau_{\geq d}V)_n$ satisfies $|\mu| < d$ by definition. The branching rule states that the S_m -irreducible representations $V(\nu)_m = V_{\nu[m]}$ which occur in the restriction of $V(\mu)_n = V_{\mu[n]}$ to S_m are exactly those for which $\nu[m] \vdash m$ can be obtained from $\mu[n] \vdash n$ by removing $n - m$ boxes. In particular, since $\mu[n]$ has $|\mu| < d$ boxes below the first row, any partition $\nu[m]$ obtained by removing boxes from $\mu[n]$ must have $< d$ boxes below the first row as well. We conclude that $V(\lambda)_m$ never occurs in this restriction, so there can be no nonzero S_m -equivariant map from $V(\lambda)_m$ to $V_n/(\tau_{\geq d}V)_n$. This demonstrates that $\tau_{\geq d}V$ is preserved by all maps f_* , and thus defines a sub-FI-module of V . \square

Remark 2.55. The definition of weight in Definition 2.50, and of the functor $\tau_{\geq d}$ in Proposition 2.54, are formulated in a way that only makes sense over a field of characteristic 0. However, it is possible to define the collection of FI-modules of weight $\leq d$ for an arbitrary ring k : namely, as the minimal collection which contains all FI-modules finitely generated in degree $\leq d$ and is closed under subquotients and extensions. We prove in [CEF] that when k is a field of characteristic 0, this definition coincides with Definition 2.50. The key to the proof is to show that the functor $\tau_{\geq d}$ can be defined abstractly for any FI-module over any ring.

Proposition 2.56 (The FI-module $V(\lambda)$). *Let k be a field of characteristic 0, and let λ be a partition. There is an FI-module $V(\lambda)$ satisfying*

$$V(\lambda)_n = \begin{cases} V_{\lambda[n]} & \text{if } n \geq |\lambda| + \lambda_1 \\ 0 & \text{otherwise} \end{cases} \quad (17)$$

that is finitely generated in degree $|\lambda| + \lambda_1$ and has stability degree $\leq \lambda_1$.

The property (17) guarantees that there is no conflict between our notation for $V(\lambda)$ and our notational convention $V(\lambda)_n := V_{\lambda[n]}$ for irreducible representations of S_n from Definition 2.8.

Proof. We define the FI-module $V(\lambda)$ as a submodule of $M(\lambda)$ by

$$V(\lambda) := \tau_{\geq |\lambda|} M(\lambda).$$

By the definition of $\tau_{\geq |\lambda|}$, the degree- n part of $V(\lambda)$ is the sum of all irreducible constituents of $M(\lambda)$ corresponding to partitions with at least $|\lambda|$ boxes beneath the top row. This can be easily computed via the branching rule: there are no such constituents unless $n \geq |\lambda| + \lambda_1$, in which case the unique such constituent is a single copy of $V_{\lambda[n]}$, verifying (17). The “monotonicity” of $M(\lambda)$ proved in [Ch, Theorem 2.8] says precisely that $V(\lambda)$ is generated in degree $|\lambda| + \lambda_1$, as claimed.

Since $V(\lambda)$ is a sub-FI-module of $M(\lambda)$, Remark 2.36 implies that $V(\lambda)$ has injectivity degree 0. Let $\mu = (\lambda_1, \lambda_1, \dots, \lambda_\ell) = \lambda[|\lambda| + \lambda_1]$. Since $V(\lambda)$ is generated by $V(\lambda)_{|\lambda| + \lambda_1} \simeq V_\mu$, we can write $V(\lambda)$ as a quotient of $M(\mu)$. By Proposition 2.41 $M(\mu)$ has surjectivity degree $\mu_1 = \lambda_1$, so Remark 2.36 implies that $V(\lambda)$ has surjectivity degree $\leq \lambda_1$. \square

2.6 Stability degree and representation stability

Our main goal in this section is to prove Proposition 2.58, relating stability degree of FI-modules with uniform representation stability in the sense of [CF]. We first quickly recall the definition of representation stability; see [CF] for a detailed treatment, and see [FH] for the representation theory facts that we use in this section.

Definition 2.57 (Representation stability). Assume that $\{V_n, \phi_n\}$ is a consistent sequence of S_n -representations, as defined preceding Lemma 2.1, over a field of characteristic 0. The sequence $\{V_n, \phi_n\}$ is *representation stable* if, for sufficiently large n , each of the following conditions holds.

I. Injectivity: The map $\phi_n: V_n \rightarrow V_{n+1}$ is injective.

II. Surjectivity: The span of the S_{n+1} -orbit of $\phi_n(V_n)$ equals all of V_{n+1} .

III. Multiplicities: Decompose V_n into irreducible representations as

$$V_n = \bigoplus_{\lambda} c_{\lambda,n} V(\lambda)_n$$

with multiplicities $0 \leq c_{\lambda,n} \leq \infty$. For each λ , the multiplicities $c_{\lambda,n}$ are eventually independent of n .

The sequence $\{V_n\}$ is *uniformly representation stable with stable range* $n \geq N$ if the multiplicities $c_{\lambda,n}$ are independent of n for all $n \geq N$, with no dependence on λ .

The following proposition shows that a stability degree for an FI-module V guarantees that it is uniformly representation stable with a stable range that can be specified quite precisely.

Proposition 2.58 (Stability degree and representation stability). *Assume that k is a field of characteristic 0. Let V be an FI-module of weight $\leq d$ with stability degree $\leq s$. Then V_n is uniformly representation stable with stable range $n \geq s + d$.*

For FI \sharp -modules in characteristic 0, we proved in Proposition 2.53 that $s \leq d$, so the following corollary is immediate from Proposition 2.58.

Corollary 2.59. *Assume that k is a field of characteristic 0. Let V be an FI \sharp -module of weight $\leq d$. Then V_n is uniformly representation stable with stable range $n \geq 2d$.*

Proof of Proposition 2.58. Let K_n be the kernel of the map $I_n: V_n \rightarrow V_{n+1}$. To prove Condition I of representation stability, we need to show that $K_n = 0$ for all $n \geq s + d$. By hypothesis, every irreducible subrepresentation $V(\lambda)_n$ of V_n , and thus also of K_n , satisfies $|\lambda| \leq d$. By Lemma 2.40(iv), it suffices to show that $(K_n)_{S_{n-d}} = 0$ for $n \geq s + d$. The composition of the two homomorphisms

$$(V_n)_{S_{n-d}} \rightarrow (V_{n+1})_{S_{n-d}} \rightarrow (V_{n+1})_{S_{n+1-d}}$$

is an isomorphism by hypothesis when $n \geq s + d$, so the first morphism is injective in this range. But the kernel of the first morphism is $(K_n)_{S_{n-d}}$, because the operation of taking S_{n-d} -coinvariants is exact in characteristic 0. We conclude that $(K_n)_{S_{n-d}} = 0$, and thus that $K_n = 0$.

Similarly, let C_{n+1} be the cokernel of the map $\text{Ind}_{S_n}^{S_{n+1}} I_n: \text{Ind}_{S_n}^{S_{n+1}} V_n \rightarrow V_{n+1}$ induced by I_n . To prove Condition II of representation stability, we need to show that $C_{n+1} = 0$ for all $n \geq s + d$. As before, it suffices to show that $(C_{n+1})_{S_{n+1-d}} = 0$. But the composition of the two homomorphisms

$$(V_n)_{S_{n-d}} \rightarrow (\text{Ind}_{S_n}^{S_{n+1}} V_n)_{S_{n+1-d}} \rightarrow (V_{n+1})_{S_{n+1-d}}$$

is an isomorphism when $n \geq s + d$, so the second homomorphism must be surjective. This implies that $(C_{n+1})_{S_{n+1-d}}$ vanishes for $n \geq s + d$. By Lemma 2.40(iv) we conclude that $C_{n+1} = 0$ for $n \geq s + d$.

It remains to prove that the multiplicity $c_{\lambda,n}$ of $V(\lambda)_n$ in V_n is constant when $n \geq s + d$. We prove this by induction on $|\lambda|$; no base case will be necessary. Note that Proposition 2.42 states that each $V(\lambda)_n$ which occurs in V_n satisfies $\lambda_1 \leq s$, so Lemmas 2.40(ii–iii) apply in our range $n \geq s + d \geq \lambda_1 + m$.

Given $m \leq d$, assume that for all μ with $|\mu| < m$ the multiplicity $c_{\mu,n}$ of $V(\mu)_n$ in V_n is constant for $n \geq s + d$. The definition of stability degree tells us that $(V_n)_{S_{n-m}}$ is independent of n for $n \geq s + m$. By Lemma 2.40(i), only those $V(\lambda)_n$ with $|\lambda| \leq m$ contribute to $(V_n)_{S_{n-m}}$, so we may write

$$(V_n)_{S_{n-m}} = \bigoplus_{|\mu| < m} c_{\mu,n}(V(\mu)_n)_{S_{n-m}} \oplus \bigoplus_{|\lambda|=m} c_{\lambda,n}(V(\lambda)_n)_{S_{n-m}} \quad (18)$$

By Lemma 2.40(iii) the factor $(V(\mu)_n)_{S_{n-m}}$ is independent of n for $n \geq s + d$. Since the left hand side of (18) and the first summand on the right are independent of n for $n \geq s + d$, the same is true of the last summand. But by Lemma 2.40(ii) the last summand of (18) is simply $\bigoplus_{|\lambda|=m} c_{\lambda,n} V_\lambda$. We conclude that for $|\lambda| = m$ the multiplicity $c_{\lambda,n}$ is constant for $n \geq s + d$, as desired. \square

2.7 The Noetherian property and representation stability

In this section we prove Theorem 2.60, and use it to prove Theorem 1.14, which relates representation stability with finite generation.

Theorem 2.60. *Let k be a Noetherian ring containing \mathbb{Q} (for example, a field of characteristic 0). Then the category of FI-modules over k is Noetherian: any sub-FI-module of a finitely generated FI-module is finitely generated.*

Proof. Let V be a finitely generated FI-module generated in degree $\leq a$, and let W be a submodule of V . The graded $k[T]$ -module $\Phi_a(V)$ is finitely generated by Proposition 2.39. Since Φ_a is exact over rings containing \mathbb{Q} , we have that $\Phi_a(W)$ is a submodule of $\Phi_a(V)$. Since $k[T]$ is a Noetherian ring, its submodule $\Phi_a(W)$ is finitely generated as a $k[T]$ -module.

Choose a finite set of generators x_1, \dots, x_r of $\Phi_a(W)$ as a $k[T]$ -module, which we can assume are homogeneous, meaning that each x_i lies in $(\Phi_a(W))_{n_i} = (W_{n_i+a})_{S_{n_i}}$ for some n_i . Since $W_{n+a} \twoheadrightarrow (W_{n+a})_{S_n}$ is surjective, we can choose elements $w_i \in W_{n_i+a}$ so that w_i projects to x_i . Let W_0 be the sub-FI-module of W generated by the lifts w_1, \dots, w_r . Since $W_0 \leq W$ we have $\Phi_a(W_0) \leq \Phi_a(W)$. Since $\Phi_a(W_0)$ contains the generating set $\{x_i\}$ we have $\Phi_a(W_0) \geq \Phi_a(W)$. Thus $\Phi_a(W_0) = \Phi_a(W)$. Since Φ_a is exact, this implies that $\Phi_a(W/W_0)$ is trivial. By Proposition 2.52 this implies that $(W/W_0)_n$ vanishes for all $n \geq a$, which is to say that W differs from W_0 in only finitely many degrees. Since V is finitely generated, Proposition 2.16 implies that V_n is a finitely-generated k -module for each n . Since k is Noetherian, this implies that its submodule W_n is finitely generated as well. We conclude that the generators w_i together with a generating set for W_n for each $n < a$ give a finite generating set for W , which proves the theorem. \square

We can now complete the proof of Theorem 1.14, which states that representation stability is equivalent to finite generation for FI-modules over a field of characteristic 0. We also observe along the way that finitely generated FI-modules are monotone in the sense of [Ch, Definition 1.2].

Proof of Theorem 1.14. Assume that V is finitely generated, so there is a surjection $\bigoplus_{i=0}^g M(i)^{\oplus b_i} \twoheadrightarrow V$ with kernel K . By Theorem 2.60, the submodule K of V is finitely generated, say in degree $\leq r$. Theorem 2.49 then implies that V has stability degree $\leq N = \max(r, g)$. Finally, Proposition 2.58 implies that $\{V_n, \phi_n\}$ is uniformly representation stable in degrees $\geq N + g$. Another application of Proposition 2.58 shows that $\{K_n\}$ is uniformly representation stable in some appropriate range, and [Ch, Theorem 2.8] states that $\{\bigoplus_{i=0}^g M(i)_n^{\oplus b_i}\}$ is monotone. It then follows from [Ch, Proposition 2.3] that $\{V_n\}$ is monotone.

It remains to show that uniform representation stability implies finite generation. Assume that $\{V_n, \phi_n\}$ is uniformly representation stable for $n \geq N$. The “surjectivity” condition states that for

$n \geq N$ the S_{n+1} -span of the image of the map $\phi_n: V_n \rightarrow V_{n+1}$ is all of V_{n+1} . Equivalently, the images of all FI-maps $V_n \rightarrow V_{n+1}$ span V_{n+1} . By induction, V is spanned by V_N together with V_k for $k < N$. Since each V_n is finite-dimensional by assumption, this shows that V is finitely generated. \square

2.8 Stability of Schur functors and Murnaghan's theorem

Tensor products. The tensor product $V \otimes W$ of two FI-modules V and W is the FI-module defined by $(V \otimes W)_n = V_n \otimes W_n$, with $f_*: (V \otimes W)_m \rightarrow (V \otimes W)_n$ acting diagonally via the maps $f_*: V_m \rightarrow V_n$ and $f_*: W_m \rightarrow W_n$.

Proposition 2.61 (Tensor products of f.g. FI-modules). *If V and W are finitely generated FI-modules, so is $V \otimes W$. If V is generated in degree $\leq m$ and W is generated in degree $\leq m'$, then $V \otimes W$ is generated in degree $\leq m + m'$. Similarly, $\text{weight}(V \otimes W) \leq \text{weight}(V) + \text{weight}(W)$.*

Proof. By Proposition 2.16 we can reduce to the case when both V and W are direct sums of modules of the form $M(m)$; note that this reduction does not increase the degree where V and W are generated. Thus it suffices to show that $X := M(m) \otimes M(m')$ is finitely generated in degree $\leq m + m'$. The space X_n has a basis given by pairs of injections $f: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ and $f': \{1, \dots, m'\} \rightarrow \{1, \dots, n\}$. Let Y be the vector space spanned by $\text{Hom}_{\text{FI}}(\mathbf{m}, \mathbf{m} + \mathbf{m}') \times \text{Hom}_{\text{FI}}(\mathbf{m}', \mathbf{m} + \mathbf{m}')$, which can be considered as a representation of $S_{m+m'}$. It is clear that, when $n \geq m + m'$, we can find a triple

$$(g, g', h) \in \text{Hom}_{\text{FI}}(\mathbf{m}, \mathbf{m} + \mathbf{m}') \times \text{Hom}_{\text{FI}}(\mathbf{m}', \mathbf{m} + \mathbf{m}') \times \text{Hom}_{\text{FI}}(\mathbf{m} + \mathbf{m}', \mathbf{n})$$

such that $f = h \circ g$ and $f' = h \circ g'$. This shows that X admits a surjection from $M(Y) \oplus \bigoplus_{k < m+m'} M(X_k)$, which completes the proof of the main assertion. The last assertion follows from the definition of weight and a well-known property of Kronecker coefficients: if $V(\nu)_n$ occurs in the tensor product $V(\lambda)_n \otimes V(\mu)_n$, then $|\nu| \leq |\lambda| + |\mu|$. \square

Stability for Schur functors. Let k be a field of characteristic 0, and let λ be a partition. The Schur functor \mathbb{S}_λ is a functor from k -vector spaces to k -vector spaces whose properties are the subject of substantial interest in combinatorics and representation theory. See [FH, Lecture 6] for the basics of Schur functors. Since \mathbb{S}_λ is a covariant functor, any FI-module V can be composed with it to give an FI-module $\mathbb{S}_\lambda V$, which satisfies $(\mathbb{S}_\lambda V)_n = \mathbb{S}_\lambda(V_n)$.

Proposition 2.62 (Finite generation of Schur functors). *Assume that $\text{char } k = 0$. Let V be a finitely generated FI-module generated in degree (resp. of weight) $\leq m$. Then for any partition λ , the FI-module $\mathbb{S}_\lambda(V)$ is finitely generated in degree (resp. of weight) $\leq m|\lambda|$.*

Proof. Let $d = |\lambda|$. Proposition 2.61 implies that $V^{\otimes d}$ is a finitely generated FI-module generated in degree (resp. of weight) $\leq md$. Note that S_d acts on $V^{\otimes d}$ by permuting the factors. Let $c_\lambda \in k[S_d]$ be the Young symmetrizer associated to the partition λ . The element c_λ defines an endomorphism of the FI-module $V^{\otimes d}$ whose image is isomorphic to $\mathbb{S}_\lambda(V)$. Thus as a quotient of $V^{\otimes d}$, the FI-module $\mathbb{S}_\lambda(V)$ is finitely generated in degree (resp. of weight) $\leq md = m|\lambda|$. \square

Applying Proposition 2.62 to the finitely generated FI-module $V(\mu)$ from Proposition 2.56, we conclude that $\mathbb{S}_\lambda(V(\mu))$ is a finitely generated FI-module. Theorem 1.14 then implies the following.

Proposition 2.63 (Schur functors of irreducibles). *Let λ, μ be partitions. There exists a finite set S of partitions ν and a set of nonnegative integers $\beta_{\lambda, \mu}^\nu$ such that for all sufficiently large n :*

$$\mathbb{S}_\lambda(V(\mu)_n) = \bigoplus_{\nu \in S} \beta_{\lambda, \mu}^\nu V(\nu)_n$$

Applying Theorem 1.14 and Proposition 2.62 to the FI-module $V = M(1)$ with $k = \mathbb{Q}$, for which $V_n = \mathbb{Q}^n$ is the permutation representation, yields the following corollary.

Corollary 2.64 (Stability of Schur functors). *The sequence of S_n -representations $\{\mathbb{S}_\lambda(\mathbb{Q}^n)\}$ is monotone in the sense of [Ch] and uniformly representation stable.*

Corollary 2.64 resolves a basic issue left open in [CF] (cf. Theorem 3.1 of [CF]). The relative simplicity of the proof given here illustrates the power of the language of FI-modules.

Murnaghan's theorem. In 1938 Murnaghan stated the following theorem; the first complete proof of this theorem was given in 1957 by Littlewood [Li].

Theorem 2.65 (Murnaghan's Theorem). *For each pair of partitions λ and μ there exists a finite set S of partitions ν and a set of nonnegative integers $g_{\lambda,\mu}^\nu$ such that for all sufficiently large n :*

$$V(\lambda)_n \otimes V(\mu)_n = \bigoplus_{\nu \in S} g_{\lambda,\mu}^\nu V(\nu)_n. \quad (19)$$

From the FI-module point of view, Murnaghan's theorem is not merely an assertion about a list of numbers. It is a structural statement about a single object, the FI-module $V(\lambda) \otimes V(\mu)$.

Proof. Since $V(\lambda)$ and $V(\mu)$ are finitely generated, Proposition 2.61 implies that $V(\lambda) \otimes V(\mu)$ is finitely generated. Thus Theorem 1.14 implies that the sequence of S_n -representations

$$(V(\lambda) \otimes V(\mu))_n = V(\lambda)_n \otimes V(\mu)_n$$

is uniformly representation stable, which implies (19). \square

2.9 Characters of FI-modules

In this section we prove Theorem 1.6. In fact, we will prove a more refined version of that theorem. Recall that for each $i \geq 1$ and any $n \geq 0$, the class function $X_i: S_n \rightarrow \mathbb{N}$ is defined by

$$X_i(\sigma) := \text{number of } i\text{-cycles in } \sigma.$$

For example, $X_1(\sigma)$ is the number of fixed points of the permutation σ . Polynomials in the variables X_i are called *character polynomials*. Class functions form a ring under pointwise product, so any character polynomial $P \in \mathbb{Q}[X_1, X_2, \dots]$ also defines a class function $P: S_n \rightarrow \mathbb{Q}$ for all $n \geq 0$. The *degree* of a character polynomial is defined by setting $\deg(X_i) = i$.

Classically, the interest in character polynomials is driven by the following fact, which seems to have been known, at least implicitly, as far back as Murnaghan; Macdonald traces it back to work of Frobenius in 1904. For a more recent reference, see Garsia–Goupil [GG].

Proposition 2.66 ([Mac, Example I.7.14]). *For every partition $\lambda \vdash d$ there is a character polynomial $P_\lambda \in \mathbb{Q}[X_1, X_2, \dots]$ of degree d such that*

$$\chi_{V(\lambda)_n}(\sigma) = P_\lambda(\sigma)$$

for all $n \geq d + \lambda_1$ and all $\sigma \in S_n$.

Recall that the range $n \geq d + \lambda_1$ is exactly the range where the notation $V(\lambda)_n$ is defined. For example, when $\lambda = \square$ we have $P_\lambda = X_1 - 1$, which agrees with the character of $V(\lambda)_n$ for all $n \geq 2$. Since all irreducible characters of S_n are integer-valued, the polynomials P_λ are integer-valued as well.

Theorem 2.67 (Polynomiality of characters). *Let k be a field of characteristic 0. Let V be a finitely generated FI-module with stability degree $\leq s$ and weight $\leq d$. Then there is a unique polynomial $P_V \in \mathbb{Q}[X_1, X_2, \dots]$ of degree at most d such that*

$$\chi_{V_n}(\sigma) = P_V(\sigma) \quad (20)$$

for all $n \geq s + d$ and all $\sigma \in S_n$. In particular, setting $F_V(n) = P_V(n, 0, \dots, 0)$ we have:

$$\dim V_n = \chi_{V_n}(\text{id}) = F_V(n) \quad (21)$$

for all $n \geq s + d$. If V is an FI \sharp -module then (20) and (21) hold for all $n \geq 0$.

Proof. We have shown in Proposition 2.58 that under the hypotheses of the theorem there is a decomposition

$$V_n \simeq \bigoplus_{\lambda} c_{\lambda} V(\lambda)_n \quad (22)$$

which holds for all $n \geq s + d$. Thus we define

$$P_V = \sum_{\lambda} c_{\lambda} P_{\lambda}$$

where P_{λ} is as in Proposition 2.66. Let λ be a partition such that $c_{\lambda} \neq 0$. Since V has weight $\leq d$ we know that $|\lambda| \leq d$ (giving the bound on the degree of P_V), and from Proposition 2.42 we know that $\lambda_1 \leq s$. It now follows from Proposition 2.66 that $\chi_n = P_V$ for all $n \geq s + d$, as desired.

We now prove the second assertion, concerning the case where V is an FI \sharp -module, by providing an explicit formula for the character polynomial P_V . The truth of the theorem is preserved by direct sums, so by Theorem 2.24 we can assume that $V = M(W)$ for some representation W of S_d . If $\lambda \vdash d$ is a partition of d , let $\chi_W(\lambda)$ denote the character of W on the conjugacy class of S_d whose cycle decomposition is encoded by λ . Let $n_i(\lambda)$ be the number of parts of λ equal to i , so that $\sum i \cdot n_i(\lambda) = |\lambda| = d$. Given such a partition $\lambda \vdash d$, define the polynomial $\binom{X_{\bullet}}{\lambda} \in \mathbb{Q}[X_1, X_2, \dots]$ by:

$$\binom{X_{\bullet}}{\lambda} := \binom{X_1}{n_1(\lambda)} \binom{X_2}{n_2(\lambda)} \cdots \binom{X_d}{n_d(\lambda)}$$

Then it is easy to check using the definition of the induced representation that for any $n \geq d$, the character of $M(W)_n$ is given by the following polynomial $P_V \in \mathbb{Q}[X_1, X_2, \dots]$:

$$P_V = \sum_{\lambda \vdash d} \chi_W(\lambda) \binom{X_{\bullet}}{\lambda}$$

Moreover, the polynomial $\binom{X_{\bullet}}{\lambda}$ clearly vanishes whenever X_i is an integer $0 \leq X_i < n_i(\lambda)$ for some i . When $n < d$, this necessarily holds for the cycle decomposition of every permutation $\sigma \in S_n$, so $P_V(\sigma) = 0$ for all $\sigma \in S_n$. Since $M(W)_n = 0$ for $n < d$, this verifies that $\chi_{M(W)_n}$ is given by P_V for all $n \geq 0$, as desired. Note that the polynomial $\binom{X_{\bullet}}{\lambda}$ has degree $\sum i \cdot n_i(\lambda) = d$, so P_V has degree d . \square

Remark 2.68 (Finite projective resolutions). Over a field of characteristic 0, Theorem 2.29 states that the projective FI-modules are exactly those that can be extended to FI \sharp -modules. So Theorem 2.67 implies that the characters of any finitely generated projective FI-module are polynomial for all $n \geq 0$. Since characters are additive in exact sequences, the same is true for any finitely generated FI-module admitting a finite projective resolution: the character χ_{V_n} is given by the character polynomial P_V for all $n \geq 0$.

So if this is *not* the case, then V does not have finite projective dimension. For example, let V be the FI-module from Remark 2.10 with $V_0 = k$ and $V_i = 0$ for each $i > 0$. This has character polynomial $P_V = 0$, but when $n = 0$, the character $\chi_{V_0}(\text{id}) = \dim V_0 = 1$ fails to agree with $P_V(0, \dots, 0) = 0$, so this FI-module has no finite projective resolution. We did however give an *infinite* projective resolution for V in (10).

2.10 FI-algebras

In this section we define FI-algebras and co-FI-algebras and establish their basic properties.

Graded FI-modules of finite type. In many applications it is useful to introduce a graded version of finite generation. For example, the sequence of rings $R_n = \{k[x_1, \dots, x_n]\}$ is not finitely generated when considered as an FI-module R , since the individual R_n are not even finite-dimensional k -vector spaces (assuming for simplicity that k is a field). However, when R is endowed with the usual grading in which each x_i has grade 1, each graded piece of R is a finitely generated FI-module. This example motivates the following definition.

Definition 2.69 (Graded FI-modules of finite type). A *graded FI-module* $V = \bigoplus V^i$ is a functor from FI to the category of graded k -modules. Note that each V^i is itself an FI-module. We say a graded FI-module V is *of finite type* if each FI-module V^i is a finitely generated FI-module. If V is supported in non-negative degrees, we say that V has *slope* $\leq m$ if each FI-module V^i has weight $\leq m \cdot i$.

Note that if V is a graded FI-module of finite type, then any quotient of V is of finite type, and over a field of characteristic 0 any submodule of V is also of finite type. If V has slope $\leq m$ then any subquotient of V has slope $\leq m$. Although a graded FI-module can be of finite type without having finite slope (if $\text{weight}(V^i)$ grows faster than linearly), we do not know of any interesting examples where this is the case.

Recall that if $U = \bigoplus U^j$ and $W = \bigoplus W^j$ are graded k -modules then $U \otimes W$ comes equipped with a natural grading $U \otimes W = \bigoplus (U \otimes W)^k$ where

$$(U \otimes W)^k = \bigoplus_{i+j=k} U^i \otimes W^j.$$

Tensor products of FI-modules of finite type. The following two propositions show that we can safely take tensor products of graded FI-modules of finite type.

Proposition 2.70. *Let U and W be graded FI-modules of finite type. If U and W are bounded below, meaning that they are supported in grades $\geq b$ for some $b \in \mathbb{Z}$, then the tensor product $U \otimes W$ is a graded FI-module of finite type. If U and W have slope $\leq m$, then $U \otimes W$ has slope $\leq m$.*

Proof. By Proposition 2.61, $U^i \otimes W^j$ is a finitely generated FI-module for each fixed i and j . The condition on the support of U and W implies that each graded piece $(U \otimes W)^k$ involves only finitely many such summands, and thus is finitely generated. The last claim follows since $\text{weight}(U^i \otimes W^j) \leq \text{weight}(U^i) \cdot \text{weight}(W^j) \leq mi + mj = mk$. \square

In Sections 3 and 4, we will need to consider a different sort of tensor product. For simplicity, we begin by considering a graded k -module V equipped with a specified injection $k \hookrightarrow V^0$. In this case we define the graded FI-module $V^{\otimes \bullet}$ to have $(V^{\otimes \bullet})_n = V^{\otimes n}$, with the inherited grading described above. The injection $f: \{1, \dots, m\} \hookrightarrow \{1, \dots, n\}$ induces a map $f_*: V^{\otimes m} \rightarrow V^{\otimes n}$ by permuting the factors

according to f , and inserting $1 \in k \subset V^0$ into all other factors. For example, the map $I_n: \{1, \dots, n\} \hookrightarrow \{1, \dots, n+1\}$ induces the map $V^{\otimes n} \rightarrow V^{\otimes n+1}$ defined by $v_1 \otimes \dots \otimes v_n \mapsto v_1 \otimes \dots \otimes v_n \otimes 1$.

If we instead specify a surjection $\eta: V \twoheadrightarrow k$ of graded k -modules, we obtain a graded co-FI-module structure on $V^{\otimes \bullet}$ by applying η to all factors not in the image of f . For example, the map $I_n: \{1, \dots, n\} \hookrightarrow \{1, \dots, n+1\}$ would then induce the map $V^{\otimes n+1} \rightarrow V^{\otimes n}$ defined by $v_1 \otimes \dots \otimes v_n \otimes v_{n+1} \mapsto \eta(v_{n+1}) \cdot v_1 \otimes \dots \otimes v_n$. Combining these, we see that a splitting $k \rightarrow V \rightarrow k$ induces the structure of a graded FI \sharp -module on $V^{\otimes \bullet}$.

Recall that $M(0)$ is the FI-module having $M(0)_n = k$ for all $n \geq 0$. If V is a graded FI-module equipped with an inclusion $M(0) \hookrightarrow V^0$, we can extend the definition of $V^{\otimes \bullet}$ above as follows.

Definition 2.71. We set $(V^{\otimes \bullet})_n = (V_n)^{\otimes n}$. The map $f_*: (V_m)^{\otimes m} \rightarrow (V_n)^{\otimes n}$ induced by the inclusion $f: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ is the composition of two maps: first, the diagonal map $(V_m)^{\otimes m} \rightarrow (V_n)^{\otimes m}$ induced by $f_*: V_m \rightarrow V_n$, and second the inclusion $(V_n)^{\otimes m} \rightarrow (V_n)^{\otimes n}$ determined by f as described in the previous paragraphs. As above, if V is a graded co-FI-module with a surjection from V to $M(0)$ in grading 0, we obtain a co-FI-module structure on $V^{\otimes \bullet}$, and if V is a graded FI \sharp -module with a splitting $M(0) \rightarrow V \rightarrow M(0)$ then $V^{\otimes \bullet}$ is a graded FI \sharp -module.

Proposition 2.72. *Let V be a graded FI-module of finite type, supported in non-negative grades and with an isomorphism $V^0 \simeq M(0)$. Then the graded FI-module $V^{\otimes \bullet}$ is of finite type. If V^i is generated in degree $\leq m \cdot i$ then $V^{\otimes \bullet}$ is generated in degree $\leq (m+1) \cdot i$.*

Proof. If we focus on grading i , we have a decomposition

$$(V^{\otimes \bullet})_n^i = ((V_n)^{\otimes n})^i \simeq \bigoplus_{i_1 + \dots + i_n = i} V_n^{i_1} \otimes \dots \otimes V_n^{i_n}.$$

The action of S_n permutes these summands and acts transitively on the collection of summands with the same multiset of indices $\{i_1, \dots, i_n\}$. In particular, we see that when $n > i$, any such summand must have some $i_\ell = 0$, and thus up to the action of S_n is of the form $V_n^{i_1} \otimes \dots \otimes V_n^{i_i} \otimes k^{\otimes n-i}$. Any inclusion $f: \{1, \dots, n\} \hookrightarrow \{1, \dots, n+1\}$ which restricts to the identity on $\{1, \dots, i\}$ induces a map from this summand to $V_{n+1}^{i_1} \otimes \dots \otimes V_{n+1}^{i_i} \otimes k^{n+1-i}$, and we would like to show that for sufficiently large n the images of these maps span $V_{n+1}^{i_1} \otimes \dots \otimes V_{n+1}^{i_i} \otimes k^{n+1-i}$.

Comparing with the definition of S_{+i} in Definition 2.30, we see that this is equivalent to the assertion that $S_{+i}(V^{i_1} \otimes \dots \otimes V^{i_i})$ is finitely generated (however there is a shift in degree associated to S_{+i} , as we discuss below). But Proposition 2.61 implies that $V^{i_1} \otimes \dots \otimes V^{i_i}$ is finitely generated, and then Proposition 2.31 implies that $S_{+i}(V^{i_1} \otimes \dots \otimes V^{i_i})$ is finitely generated. If we assume that V^j is generated in degree $\leq mj$ for all j , Proposition 2.61 implies that $V^{i_1} \otimes \dots \otimes V^{i_i}$ is generated in degree $\leq mi_1 + \dots + mi_i = mi$, and then Proposition 2.31 shows that $S_{+i}(V^{i_1} \otimes \dots \otimes V^{i_i})$ is generated in degree $\leq mi$. This yields the desired claim that $V^{\otimes \bullet}$ is generated in degree $\leq mi + i$, where the discrepancy is because we have not actually shifted our indexing here as we did in Definition 2.30. \square

FI-algebras. A *graded FI-algebra* $A = \bigoplus A^i$ is a functor from FI to the category of graded k -algebras—in other words, it is an FI-module with a graded associative k -algebra structure respected by the FI-maps. A graded FI-algebra is of finite type if the underlying graded FI-module is of finite type. Given an sub-FI-module V of A , we say that A is *generated* by V if V_n generates A_n as a k -algebra for all $n \geq 0$.

Finally, let $k\langle V \rangle = \bigoplus_{j=0}^{\infty} V^{\otimes j}$ denote the free associative algebra on V . If V is a graded k -module, the grading on $V^{\otimes j}$ described above induces a grading on $k\langle V \rangle$, so we can think of $k\langle - \rangle$ as a functor from graded k -modules to graded k -algebras. Thus if V is a graded FI-module, we can compose with this functor to obtain a new graded FI-algebra $k\langle V \rangle$.

Proposition 2.73. *Let V be a graded FI-module of finite type, supported in positive grades. Then the graded FI-algebra $k\langle V \rangle$ is of finite type. If V has slope $\leq m$, then $k\langle V \rangle$ has slope $\leq m$.*

Proof. We have:

$$k\langle V \rangle = \bigoplus_{j=0}^{\infty} V^{\otimes j}$$

By Proposition 2.70, each summand $V^{\otimes j}$ is of finite type. Moreover, since V is supported in positive grades, the summand $V^{\otimes j}$ is supported in grades $\geq j$. It follows that for any fixed grading i , the sum describing $k\langle V \rangle^i$ is finite. We conclude that $k\langle V \rangle^i$ is a finitely generated FI-module, and thus that $k\langle V \rangle$ is a graded FI-algebra of finite type. For the last claim, simply note that if V has slope $\leq m$, each summand $V^{\otimes j}$ has slope $\leq m$ by Proposition 2.70. \square

Let A be a graded FI-algebra, and let V be a graded sub-FI-module of A supported in positive grades. Then there is a natural homomorphism of FI-algebras $k\langle V \rangle \rightarrow A$. To say that A is generated by V is just to say this homomorphism is surjective.

Theorem 2.74 (Algebras generated by finite type FI-modules). *Let V be a graded FI-module of finite type supported in positive grades, and let A be a graded FI-algebra generated by V . Then A is of finite type. If V has slope $\leq m$, then A has slope $\leq m$.*

Proof. This is an immediate corollary of Proposition 2.73, since A is a quotient of $k\langle V \rangle$, which is of finite type with slope $\leq m$. \square

Corollary 2.75. *If a graded FI-algebra A is generated by a sub-FI-module V concentrated in grade 1, then A has slope $\leq \text{weight}(V)$. If V is concentrated in grade $d > 0$, then A has slope $\leq \text{weight}(V)/d$.*

In Section 3.1 we will often want to consider the FI-algebra “presented” by a collection of relations. To formalize this, we need the notion of an FI-ideal.

Definition 2.76 (FI-ideal). Let A be a graded FI-algebra. An FI-ideal I of A is a graded sub-FI-module of A such that I_n is a homogeneous ideal in the algebra A_n for each n . Equivalently, each I_n is an S_n -invariant homogeneous ideal in A_n , and for each $m < n$ the ring homomorphisms $A_m \rightarrow A_n$ induced by $\text{Hom}_{\text{FI}}(\mathbf{m}, \mathbf{n})$ carry I_m to I_n .

If I is any FI-ideal of a graded FI-algebra A , then the quotient A/I is a graded FI-algebra. Moreover, if A is of finite type, then so is A/I , and if A has slope $\leq m$, then so does A/I . This applies not only to each of the FI-algebras

$$\begin{aligned} k\langle V \rangle &:= \text{the free associative algebra on } V \\ k[V] &:= \text{the free commutative algebra on } V \\ \mathcal{L}(V) &:= \text{the free Lie algebra on } V \\ \Gamma[V] &:= \text{the free graded-commutative } k\text{-algebra on } V \end{aligned}$$

but also to any of their quotients by an FI-ideal. When V is a graded FI-module of finite type supported in positive grades, all of these FI-algebras are of finite type. Many of the applications in the second half of this paper will follow directly from these results.

Co-FI-algebras. Recall that a co-FI-module is a functor from co-FI to vector spaces. We similarly define a graded co-FI-module to be a functor from co-FI to graded k -modules, and define a graded co-FI-algebra to be a functor from co-FI to graded k -algebras. When k is a field, we say that a graded co-FI-module over k is *of finite type* (resp. *has slope $\leq m$*) if its dual is a graded FI-module of finite type (resp. *has slope $\leq m$*).

Proposition 2.77. *Assume that k is a field of characteristic 0, and suppose A is a graded co-FI-algebra. Let V be a graded co-FI-module of finite type contained in A which is supported in positive grades. Then the subalgebra of A generated by V is a graded co-FI-algebra of finite type. If V has slope $\leq m$, then the subalgebra of A generated by V has slope $\leq m$.*

Proof. Let B be the subalgebra of A generated by V , so that B is a quotient of the free associative algebra $k\langle V \rangle$. The dual B^* is thus a graded sub-FI-module of $k\langle V \rangle^*$. Each graded piece of $k\langle V \rangle^*$ is isomorphic to a tensor power of V^* , and is thus a finitely generated FI-module. Applying Theorem 2.60, we conclude that each graded component $(B^*)_n = (B_n)^*$ is a finitely generated FI-module, as desired. The last claim is proved identically. \square

Like FI-algebras, co-FI-algebras can be presented by a collection of relations. Note however that in this case we do not have bounds on the degree where the quotient is generated, since we depend on the Noetherian property to deduce finite generation.

Definition 2.78 (co-FI-ideal). Let A be a graded co-FI-algebra. A co-FI-ideal I of A is a graded sub-co-FI-module of A such that I_n is a homogeneous ideal in the algebra A_n for each n . Equivalently, each I_n is an S_n -invariant homogeneous ideal in A_n , and for each $m < n$ the ring homomorphisms $A_n \rightarrow A_m$ induced by $\text{Hom}_{\text{co-FI}}(\mathbf{n}, \mathbf{m})$ carry I_n to I_m .

Proposition 2.79. *Assume that k is a field of characteristic 0. If A is a graded co-FI-algebra of finite type, and I is a co-FI-ideal in A , then the quotient A/I is a co-FI-algebra of finite type.*

Proof. The dual $(A/I)^*$ is a sub-FI-module of A^* , which is a graded FI-module of finite type by assumption. Theorem 2.60 then implies that $(A/I)^*$ is of finite type, as desired. \square

(Co)homology of (co-)FI-spaces. One natural source of FI-algebras is the cohomology rings of co-FI-spaces, or the duals of the cohomology rings of FI-spaces. In that context, the following is an immediate corollary of the results above.

Corollary 2.80 (Cohomology algebras). *Let X be a co-FI-space, and suppose that S is a set of nonnegative integers such that the cohomology $H^s(X; k)$ is a finitely generated FI-module for each $s \in S$. Then the subalgebra of $H^*(X; k)$ generated by $H^s(X; k)$ for all $s \in S$ is a graded FI-algebra of finite type.*

If X is instead an FI-space, assume that k is a field of characteristic 0, and $H_s(X; k)$ is a finitely generated FI-module for each $s \in S$. Then the subalgebra of $H^(X; k)$ generated by $H^s(X; k)$ for all $s \in S$ is a graded co-FI-algebra of finite type.*

3 Applications: Coinvariant algebras and polynomials on rank varieties

In this section we apply the theory of FI-modules to obtain theorems in algebraic combinatorics and algebraic geometry.

3.1 Algebras with explicit presentations

One way that FI-algebras naturally arise is from explicit presentations by generators and relations.

Example 3.1 (Nilpotent rings). Begin with the graded FI-algebra $k[M(1)]$, whose n^{th} algebra is $k[M(1)]_n = k[x_1, \dots, x_n]$. For each fixed $d \geq 2$ the sequence of nilpotent rings

$$\{k[x_1, \dots, x_n]/(x_1^d, \dots, x_n^d)\}$$

forms a graded FI-algebra of finite type by Theorem 2.74. Thus by Theorem 1.14 the i -th graded piece of this algebra is uniformly representation stable for all $i \geq 0$. When $d = 2$, this fact was first proved by Ashraf–Azam–Berceneau as Corollary 5.2 in [AAB].

Example 3.2 (Symmetrically presented algebras). Let $P_1(x_1, \dots, x_{m_1}), \dots, P_r(x_1, \dots, x_{m_r})$ be a finite collection of polynomials, and consider the FI-ideal I generated by the polynomials P_1, \dots, P_r . Concretely, let I_n be the ideal of $k[x_1, \dots, x_n]$ generated by the finite set

$$\bigcup_{i=1}^r \bigcup_f \{P_i(x_{f(1)}, \dots, x_{f(m_i)})\}$$

as f ranges over the injections of $\{1, \dots, m_i\}$ into $\{1, \dots, n\}$. By construction the $\{I_n\}$ form an FI-ideal I inside $k[M(1)]$, and so $\{k[x_1, \dots, x_n]/I_n\}$ is a graded FI-algebra of finite type. Example 3.1 is the special case where $r = m = 1$ and $P_1(x_1) = x_1^d$.

Example 3.3 (The Arnol'd algebra). Braid groups were first studied by Artin and Hurwitz over a century ago. In this section we focus on the *pure braid group* P_n , which is the fundamental group of the configuration space of n distinct complex numbers (z_1, \dots, z_n) with $z_i \in \mathbb{C}$ and $z_i \neq z_j$. This space is aspherical, so its cohomology coincides with $H^*(P_n; \mathbb{Q})$. These cohomology algebras $H^*(P_n; \mathbb{Q})$ fit together into a graded FI \sharp -algebra $H^*(P_\bullet; \mathbb{Q})$; this follows from a general result that we will prove below in Theorem 4.7. In this section we combine this FI \sharp -algebra structure with a presentation of $H^*(P_n; \mathbb{Q})$ due to Arnol'd [Ar] to describe more precisely the cohomology of the pure braid groups.

The first cohomology $V = H^1(P_\bullet; \mathbb{Q})$ is isomorphic to $V := \text{Sym}^2 M(1)/M(1)$. Concretely, a basis for V_n is given by the symbols $\{w_{ij} \mid 1 \leq i, j \leq n\}$ modulo the relations $w_{ij} = w_{ji}$ and $w_{ii} = 0$, and a partial injection f acts by $f_* w_{ij} = w_{f(i), f(j)}$ when i and j lie in the domain of f and $f_* w_{ij} = 0$ otherwise. Since $M(1)$ is generated in degree 1, Proposition 2.61 implies that $V = \text{Sym}^2 M(1)/M(1)$ is generated in degree ≤ 2 . It is immediate that $V_0 = 0$ and $V_1 = 0$, and we can easily compute $V_2 \simeq V_{\square\square}$, so we conclude that $V = \text{Sym}^2 M(1)/M(1) \simeq M(\square\square)$.

Arnol'd [Ar] proved that the cohomology ring $H^*(P_n; \mathbb{Q})$ is generated by $H^1(P_n; \mathbb{Q})$, so Theorem 2.74 implies that the graded FI \sharp -algebra $H^*(P_\bullet; \mathbb{Q})$ is of finite type. Since $V = H^1(P_\bullet; \mathbb{Q})$ is generated in degree 2, Corollary 2.75 implies that $H^*(P_\bullet; \mathbb{Q})$ has slope 2, and that $H^i(P_\bullet; \mathbb{Q})$ is generated in degree $\leq 2i$. Corollary 2.59 thus implies that for each $i \geq 0$, the sequence $\{H^i(P_n; \mathbb{Q})\}$ of S_n -representations is uniform representation stable with stable range $4i$. This yields a new proof of one of the main theorems (Theorem 4.1) in [CF].

Computing the character of $H^i(P_n; \mathbb{Q})$. Consider V as a graded FI-module by placing it in grade 1, so that $\Gamma[V]$ is a graded-commutative FI-algebra of finite type. Let I be the FI \sharp -ideal in $\Gamma[V]$ generated by

$$\{w_{jk}w_{kl} + w_{kl}w_{lj} + w_{lj}w_{jk} \mid j \neq k \neq l\} \quad (23)$$

The work of Arnol'd shows that $H^*(P_\bullet; \mathbb{Q})$ is isomorphic to the graded FI \sharp -algebra $\Gamma[V]/I$. Thanks to the classification of FI \sharp -modules, this reduces the description of $H^i(P_n; \mathbb{Q})$ to a finite computation for each $i \geq 0$. For example, we have

$$H^2(P_n; \mathbb{Q}) \simeq \bigwedge^2 H^1(P_n; \mathbb{Q})/I^{(2)}.$$

where the $I^{(2)}$ denotes the grade-2 component of I , which is an $\text{FI}\sharp$ -module. We computed in (12) that

$$M(\square\square) \otimes M(\square\square) = M(\square\square\square\square) \oplus M(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}) \oplus M(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}) \oplus M(\square\square\square) \oplus M(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix})^{\oplus 2} \oplus M(\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}) \oplus M(\square\square).$$

Exchanging the two factors gives an isomorphism from $M(\square\square) \otimes M(\square\square)$ to itself. As an isomorphism of $\text{FI}\sharp$ -modules, it necessarily preserves each of the summands in the decomposition above. By examining the details of the computation of (12) we can check that this involution acts trivially on $M(\square\square\square\square)$, $M(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix})$, $M(\square\square\square)$, and $M(\square\square)$; it acts by negation on $M(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix})$ and $M(\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix})$; and it exchanges the two $M(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix})$ summands. We conclude that $\bigwedge^2 M(\square\square) = M(\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}) \oplus M(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}) \oplus M(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix})$.

The ideal $I_n^{(2)}$ vanishes for $n < 3$ by definition, and for $n = 3$ the subspace (23) is just the alternating S_3 -representation $I_3^{(2)} \simeq V_{\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}}$. Since $I^{(2)}$ is generated in degree ≤ 3 , by Theorem 2.24 this shows that $I^{(2)} = M(\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix})$. This yields the description of $H^2(P_n; \mathbb{Q})$ mentioned in the introduction

$$\begin{aligned} H^2(P_n; \mathbb{Q}) &\simeq \bigwedge^2 M(\square\square) / M(\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}) \\ &\simeq M(\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}) \oplus M(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}) \oplus M(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}) / M(\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}) \\ &\simeq M(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}) \oplus M(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}) \end{aligned}$$

which specifies $H^2(P_n; \mathbb{Q})$ for all n simultaneously. Based on computer calculations, John Wiltshire-Gordon (personal communication) has formulated a precise conjecture for the decomposition of $H^i(P_n; \mathbb{Q})$ as a sum of $\text{FI}\sharp$ -modules $M(\lambda)$ for all $i \geq 0$.

3.2 Multivariate diagonal coinvariant algebras

Let k be a field of characteristic 0.

Classical coinvariant algebras. The *symmetric polynomials* are the S_n -invariants in the ring of polynomials $k[x_1, \dots, x_n]$, where S_n acts by permuting the variables. Let I_n be the ideal in $k[x_1, \dots, x_n]$ generated by those symmetric polynomials with vanishing constant term. The classical *coinvariant algebra* $R(n)$ is the quotient $R(n) := k[x_1, \dots, x_n] / I_n$. Chevalley [Che, Theorem B] proved that $R(n)$ is isomorphic as an S_n -representation to the regular representation $k[S_n]$.

Theorem 2.74 does not apply to the coinvariant algebras $R(n)$ since the ideals I_n do not together form an FI -ideal. For example, the symmetric polynomial $x_1 + \dots + x_n \in I_n \subset k[x_1, \dots, x_n]$ is no longer in the ideal of symmetric polynomials I_{n+1} when considered as an element of $k[x_1, \dots, x_n, x_{n+1}]$. Fortunately, we can understand the coinvariant algebra by instead using the natural *co-FI-module* structure on the algebra of polynomials.

Since $M(1)$ is an $\text{FI}\sharp$ -module, $k[M(1)]$ also carries a co- FI -module structure, which we can describe explicitly as follows. Let $f: \{1, \dots, m\} \hookrightarrow \{1, \dots, n\}$ be a map in $\text{Hom}_{\text{co-FI}}(\mathbf{n}, \mathbf{m})$. Then the corresponding algebra homomorphism $f_*: k[x_1, \dots, x_n] \rightarrow k[x_1, \dots, x_m]$ sends $x_{f(i)}$ to x_i for all $i \in \{1, \dots, m\}$, and sends x_j to 0 whenever j is not in the image of f . It is easy to see that f_* maps symmetric functions to symmetric functions, and thus that $f_*(I_n) \subseteq I_m$. We see that $I = \{I_n\}$ forms a co- FI -ideal, making the coinvariant algebras $R(n) = k[x_1, \dots, x_n] / I_n$ into a graded co- FI -algebra R .

Note that R is generated by its degree 1 part (the co- FI -module spanned by the monomials x_1, \dots, x_n , isomorphic to k^n/k for $n \geq 1$). Thus we conclude from Proposition 2.77 that R is a graded co- FI -module of finite type. In particular, Theorem 1.14 implies that the graded pieces of the

coinvariant algebra are uniformly representation stable. A different proof of this result was given in [CF, Theorem 7.4], using work of Stanley, Lusztig, and Kraskiewicz–Weyman.

Multivariate diagonal coinvariant algebras. Recall from the introduction the definition of the r -fold multivariate diagonal coinvariant algebra $R^{(r)}(n)$: we start with the free polynomial algebra in r sets of n variables

$$k[\mathbf{X}^{(r)}(n)] := k[x_1^{(1)}, \dots, x_n^{(1)}, \dots, x_1^{(r)}, \dots, x_n^{(r)}]$$

and take its quotient by the ideal I_n generated by all S_n -invariant functions with zero constant term:

$$R^{(r)}(n) = k[\mathbf{X}^{(r)}(n)]/I_n.$$

Recall furthermore that this ring comes equipped with a natural multigrading by r -tuples $J = (j_1, \dots, j_r)$ of non-negative integers, where j_k records the total degree in the variables $x_1^{(k)}, \dots, x_n^{(k)}$.

Theorem 3.4 (Diagonal coinvariant algebras are of finite type). *Assume that k is a field of characteristic 0, and fix any $r \geq 1$. Then $R^{(r)}$ is a graded co-FI-algebra of finite type. For each $J = (j_1, \dots, j_r)$ the weight of $[R_J^{(r)}]^*$ is at most $|J| = j_1 + \dots + j_r$.*

Proof. The co-FI-algebra structure on $k[M(1)^{\oplus r}] = k[\mathbf{X}^{(r)}]$ is described as follows. An injection $f: \{1, \dots, m\} \hookrightarrow \{1, \dots, n\}$ induces the map $f_*: k[\mathbf{X}^{(r)}(n)] \rightarrow k[\mathbf{X}^{(r)}(m)]$ sending $x_{f(i)}^{(j)}$ to $x_i^{(j)}$, and annihilating $x_l^{(j)}$ for all l outside the image of f . As above, these maps preserve the S_n -invariant elements in $k[\mathbf{X}^{(r)}(n)]$, and thus the ideals I_n they generate. Thus the I_n form a co-FI-ideal $I \subset k[\mathbf{X}^{(r)}]$. It follows that $R^{(r)} := k[\mathbf{X}^{(r)}]/I$ is a co-FI-algebra.

Since $k[\mathbf{X}^{(r)}]$ is generated as an algebra by its degree 1 part $M(1)^{\oplus r}$, the same is true of its quotient $R^{(r)}$. Thus by Proposition 2.77, $R^{(r)}$ is a graded co-FI-algebra of finite type. In particular, the dual of each graded piece $R_J^{(r)}$ is a finitely generated FI-module. The piece of $k[\mathbf{X}^{(r)}]$ in grading J is isomorphic as an S_n -representation to $\text{Sym}^{j_1} k^n \otimes \dots \otimes \text{Sym}^{j_r} k^n$. By Proposition 2.61 this has weight at most $|J|$. Since $[R_J^{(r)}]^*$ is a sub-FI-module of $k[\mathbf{X}^{(r)}]_J^*$ the second statement follows. \square

Recall that over a field of characteristic 0 each representation of S_n is self-dual. Thus applying Theorem 2.67 to Theorem 3.4 proves that for each $r \geq 1$ and each J , the character of the S_n -representation $R_J^{(r)}(n)$ is eventually a polynomial in $X_1, \dots, X_{|J|}$ of degree $\leq |J|$, as claimed in Theorem 1.12.

3.3 The Bhargava–Satriano algebra

In [BS] Bhargava and Satriano develop a notion of *Galois closure* for an arbitrary finite-dimensional algebra R over a field k , which reduces to the usual notion when R is a field extension (or even an étale algebra). The algebra $G(R/k)$ is a quotient of $R^{\otimes d}$, where $d = \dim_k R$, and the natural action of S_d on $R^{\otimes d}$ descends to an action of S_d on $G(R/k)$ by “Galois automorphisms”, just as in the classical case.

In many ways, the most interesting case is the most degenerate: consider the ring

$$R_n := k[x_1, \dots, x_n]/(x_i x_j)_{i,j \in \{1, \dots, n\}}$$

which has dimension $n + 1$ over k and which carries an action of S_n by permutation of the variables. We call the Galois closure $G(R_n/k)$ the *Bhargava–Satriano algebra*. Bhargava–Satriano show [BS, Theorem 7] that this ring is “maximally degenerate” among rings of the same dimension, in the sense that $\dim G(R_n/k) \geq \dim G(T_n/k)$ for any other algebra T_n with $\dim T_n = \dim R_n$.

The ring $G(R_n/k)$ carries an action of $S_n \times S_{n+1}$, the first factor coming from the action of S_n on R_n by field automorphisms, the second factor coming from the Galois automorphisms. Bhargava–Satriano compute the decomposition of $G(R_n/\mathbb{Q})$ as a representation of the Galois group S_{n+1} , and use this to show that $\dim G(R_n/\mathbb{Q}) > (n+1)!$ for $n \geq 3$. The decomposition of $G(R_n/\mathbb{Q})$ as an $S_n \times S_{n+1}$ -representation is unknown.

Proposition 3.5. *Assume that k is a field of characteristic 0. There is a graded co-FI-algebra BS of finite type satisfying $BS_n \simeq G(R_n/k)$ as graded S_n -representations, where the S_n action is given by the diagonal action of $S_n \subset S_n \times S_{n+1}$.*

Proof. The ring R_n can be thought of as a graded co-FI-algebra R of finite type, supported in grades 0 and 1; specifically, we have $R^0 = M(0)$ and $R^1 = M(1)$. Of course R_n is also an FI \sharp -algebra; however, we will see shortly that only the co-FI-algebra structure will descend to $G(R_n/k)$.

The Galois closure $G(R_n/k)$ is defined as the quotient of $R_n^{\otimes n+1}$ by certain relations, and Bhargava–Satriano show [BS, §11.2] in this case that the ideal I_n of relations in $R_n^{\otimes n+1}$ is generated by the elements:

$$\gamma_n(x_i) = x_i \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes x_i \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes x_i$$

Recall from Definition 2.71 that $R^{\otimes \bullet}$ is the graded co-FI-algebra having $(R^{\otimes \bullet})_n = R_n^{\otimes n}$. In particular, we can consider $\gamma_n(x_i)$ as an element of $(R \otimes R^{\otimes \bullet})_n$. Recalling from Definition 2.71 the co-FI-structure on $R^{\otimes \bullet}$, we see that under any co-FI-map $f \in \text{Hom}_{\text{co-FI}}(\mathbf{n}, \mathbf{m})$ we have $f_* \gamma_n(x_i) = \gamma_m(f^{-1}(x_i))$ when i is in the image of f , and 0 otherwise.

In particular, the ideals I_n generated by the $\gamma_n(x_i)$ fit together to give a co-FI-ideal $I \subset R \otimes R^{\otimes \bullet}$. We define the graded co-FI-algebra BS to be the quotient $BS := R \otimes R^{\otimes \bullet} / I$, and we have

$$G(R_n/k) \simeq BS_n \simeq R_n^{\otimes n+1} / I_n.$$

Since R is a graded co-FI-algebra of finite type, Proposition 2.72 implies that $R^{\otimes \bullet}$ is of finite type, and Proposition 2.70 implies that $R \otimes R^{\otimes \bullet}$ is as well. Applying Proposition 2.79, we conclude that the quotient BS is a graded co-FI-algebra of finite type, as desired. \square

The graded pieces BS^i can be computed for small i by hand. For instance, one checks that the dual of BS^1 is isomorphic as an FI-module to

$$(BS^1)^* \simeq M(1) \otimes M(1) = M(\square\square) \oplus M(\begin{smallmatrix} \square \\ \square \end{smallmatrix}) \oplus M(\square).$$

This raises the question: does the co-FI-algebra BS in fact have the structure of an FI \sharp -algebra?

3.4 Polynomials on rank varieties

In this section we apply our theory to a natural class of algebraic varieties. Let k be a field of arbitrary characteristic. For $n \geq 1$ let $\{x_{ij}\}$ be the coordinates on the space $\text{Mat}_{n \times n}(k)$ of $n \times n$ matrices over k . Fix $m \geq 1$. Let $P = (p_1, \dots, p_m)$ be an m -tuple of polynomials $p_i \in k[T]$, and let $r = (r_1, \dots, r_m)$ be an m -tuple of positive integers. With this data Eisenbud–Saltman [ES] define the *rank variety* of matrices:

$$X_{P,r}(n) := \{A \in \text{Mat}_{n \times n}(k) \mid \text{rank}(p_i(A)) \leq r_i, 1 \leq i \leq m\}$$

The special case when $m = 1$ and $p_1(T) = T$ is the fundamental example of the *determinantal variety* of matrices of rank $\leq r_1$. Note that $X_{P,r}(n)$ is an affine variety, defined by the polynomial equations

$$\{\det(B(p_i(A))) = 0 \mid B \in \mathcal{B}_i, 1 \leq i \leq m\}$$

where \mathcal{B}_i denotes the set of $(r_i + 1) \times (r_i + 1)$ minors of the matrix (x_{ij}) . Let I_n be the ideal of $k[\{x_{ij}\}]$ generated by these polynomials. The coordinate ring $\mathcal{O}(X_{P,r}(n))$ is isomorphic to $k[\{x_{ij}\}]/I_n$. Since the polynomials $\det(B(p_i(A)))$ are homogeneous, the grading of $k[\{x_{ij}\}]$ into homogeneous polynomials of degree $d \geq 0$ descends to a grading of $k[\{x_{ij}\}]/I_n$. A basic question to ask about the $X_{P,r}(n)$ is the following.

Question 3.6. *What is the dimension of the space of degree $d \geq 1$ polynomials on $X_{P,r}(n)$; that is, what is the dimension of the degree $d \geq 1$ component of $k[\{x_{ij}\}]/I_n$?*

In the special case of determinantal varieties in characteristic 0, an answer to Question 3.6 can be deduced from work of Lascoux and Weyman (see Corollary 6.1.5(d) of [Wey]). While we cannot resolve Question 3.6 completely, the theory of FI \sharp -modules imposes strong constraints on the answer.

Theorem 3.7 (Polynomiality for rank varieties). *Fix P , r , and d . Then the dimension of the space $\mathcal{O}(X_{P,r}(n))_d$ of degree- d polynomials on the rank variety $X_{P,r}(n)$ is polynomial in n of degree at most $2d$ for all $n \geq 0$. If $\text{char } k = 0$ the characters $\chi_{\mathcal{O}(X_{P,r}(n))_d}$ are polynomial of degree at most $2d$ for all $n \geq 0$.*

Proof. The rank varieties $X_{P,r}(n)$ fit together into an FI \sharp -scheme $X_{P,r}$. For any morphism $f = (A, B, \phi) \in \text{Hom}_{\text{FI}\sharp}(\mathbf{m}, \mathbf{n})$, the map $f_*: X_{P,r}(n) \rightarrow X_{P,r}(m)$ is given in coordinates by $x_{ij} \mapsto x_{\phi^{-1}(i), \phi^{-1}(j)}$ for $i, j \in B$ and $x_{ij} \mapsto 0$ if either i or j does not lie in B . Thus the ring of functions $\mathcal{O}(X_{P,r})$ is a graded FI \sharp -algebra. $\mathcal{O}(X_{P,r})$ is a quotient of $k[M(1) \otimes M(1)]$, and $M(1) \otimes M(1)$ is generated in degree 2; thus Theorem 2.74 implies that $\mathcal{O}(X_{P,r}(n))$ is a graded FI \sharp -algebra of finite type and slope ≤ 2 . Thus for each $d \geq 0$ the FI \sharp -module $\mathcal{O}(X_{P,r}(n))_d$ is finitely generated in degree $\leq 2d$. The theorem now follows from Theorem 2.67 and Corollary 2.27. \square

Remark 3.8. The rank variety $X_{P,r}(n)$ of Theorem 3.7 may be non-reduced as a scheme. However, the same theorem holds for the underlying reduced variety. The proof amounts to replacing the ideal I_n by its radical and applying the same argument.

Recent work of Draisma and Kuttler [DK] proves a bounded-generation result for “border rank varieties”, which generalize the rank varieties discussed here to the case of tensors of rank higher than 2. The key ingredient of their theorem is a Noetherian-type property. It would be interesting to understand whether their work can be phrased in the language of FI-modules.

4 Applications: cohomology of configuration spaces

In this section we apply the theory of FI-modules to the cohomology of configuration spaces of distinct points in a manifold M . We obtain in Theorem 4.3 and Corollary 4.5 strengthenings of earlier results of Church [Ch, Theorems 1 and 5]. The use of FI-modules serves to eliminate some of the combinatorial and notational complications that arose in [Ch]. When M is an open manifold, we apply the theory of FI \sharp -modules to obtain a number of new theorems on the rational, integral, and mod- p cohomology of the configuration spaces of M .

4.1 Finite generation

For any finite set S and any space M , let $\text{Conf}_S(M)$ denote the configuration space of distinct points on M labeled by S , which we identify with the space of embeddings of S into M :

$$\text{Conf}_S(M) := \text{Emb}(S, M)$$

In particular, $\text{Conf}_{\{1,\dots,n\}}(M)$, which for simplicity we denote $\text{Conf}_n(M)$, is the configuration space of *ordered* n -tuples of distinct points on M :

$$\text{Conf}_n(M) = \{(x_1, \dots, x_n) \in M^n \mid x_i \neq x_j\}$$

The symmetric group S_n acts on both $\text{Conf}_n(M)$ and M^n by permuting coordinates, and the inclusion $\text{Conf}_n(M) \hookrightarrow M^n$ is S_n -equivariant. The configuration spaces $\text{Conf}_n(M)$, taken together for all n , form a co-FI-space, denoted $\text{Conf}(M)$, where an inclusion $f: \{1, \dots, m\} \hookrightarrow \{1, \dots, n\}$ induces the map $f_*: \text{Conf}_n(M) \rightarrow \text{Conf}_m(M)$ defined by $f_*(x_1, \dots, x_n) = (x_{f(1)}, \dots, x_{f(m)})$. In terms of embeddings, an injection $f: S \rightarrow T$ induces a map

$$f_*: \text{Conf}_T(M) = \text{Emb}(T, M) \rightarrow \text{Emb}(S, M) = \text{Conf}_S(M)$$

by precomposing an embedding $T \rightarrow M$ with $f: S \rightarrow T$. The same formula turns the collection of products M^n into a co-FI-space M^\bullet , and the inclusions $\text{Conf}_n(M) \hookrightarrow M^n$ induce a map $\text{Conf}(M) \rightarrow M^\bullet$ of co-FI-spaces. The following theorem is proved (in different language) in [Ch].

Theorem 4.1 (Rational cohomology of configuration spaces). *Let M be a connected, oriented manifold of dimension at least 2 with $\dim_{\mathbb{Q}}(H^*(M; \mathbb{Q})) < \infty$. Then $H^*(\text{Conf}(M); \mathbb{Q})$ is a graded FI-algebra of finite type.*

Note that the condition $\dim_{\mathbb{Q}}(H^*(M; \mathbb{Q})) < \infty$ is satisfied whenever M is compact.

Proof. The requirement that $\dim M \geq 2$ ensures that the configuration space $\text{Conf}_n(M)$ is connected. Totaro [To] provides an explicit description of the E_2 page of the Leray spectral sequence of the inclusion $\text{Conf}_n(M) \hookrightarrow M^n$. More precisely, [To, Theorem 1] identifies the E_2 page with a quotient of the graded-commutative algebra

$$H^*(M^n; \mathbb{Q})[\{G_{ij}\}]$$

by certain explicit families of relations which include $G_{ii} = 0$ and $G_{ij} = (-1)^d G_{ji}$ for all $i, j \in \{1, \dots, n\}$. A bigrading on this algebra is specified by placing the generator G_{ij} in bigrading $(0, d-1)$, and $H^p(M^n; \mathbb{Q})$ in bigrading $(p, 0)$. Then E_2^{pq} is precisely the (p, q) -bigraded piece of Totaro's algebra. In particular, we note that all rows with q not divisible by $d-1$ are zero.

Functoriality of the Leray spectral sequence implies that the spectral sequence of FI-modules $E_r := E_r(\text{Conf}(M) \rightarrow M^\bullet)$ converges to the graded FI-module $H^*(\text{Conf}(M); \mathbb{Q})$. Our first goal is to prove that the graded FI-module $E_2(\text{Conf}(M) \rightarrow M^\bullet)$ is of finite type.

Since M^\bullet is a co-FI-space, its cohomology $H^*(M^\bullet; \mathbb{Q})$ is a graded FI-module. The Künneth theorem implies that $H^*(M^\bullet; \mathbb{Q})$ almost coincides with the FI-module $H^*(M)^{\otimes \bullet}$ from Definition 2.71. The only difference is that when the factors of

$$H^*(M^n; \mathbb{Q}) \simeq H^*(M; \mathbb{Q})^{\otimes n} \simeq \bigoplus H^{i_1}(M; \mathbb{Q}) \otimes \dots \otimes H^{i_n}(M; \mathbb{Q})$$

are permuted, a sign is introduced depending on the grading. But it is easy to see from the proof of Proposition 2.72 that this sign does not affect the proof of that proposition. Since M is connected, we have $H^0(M^\bullet; \mathbb{Q}) \simeq M(0)$, which verifies the remaining hypothesis of Proposition 2.72. We conclude $H^*(M^\bullet; \mathbb{Q})$ is a graded FI-module of finite type and slope $\leq i$.

It follows from an examination of Totaro's relations that the elements G_{ij} span an FI-module G placed in bidegree $(0, d-1)$ which is isomorphic to $M(\square)$ or $M(\square)$ according as d is even or odd. In either case, G is finitely generated, so Proposition 2.70 shows that $H^*(M^\bullet; \mathbb{Q}) \otimes G$ is of finite type. Since $E_2(\text{Conf}_n(M) \rightarrow M^n)$ is generated as an algebra by $H^*(M^n; \mathbb{Q})$ and G , the FI-algebra $E_2 = E_2(\text{Conf}(M) \rightarrow M^\bullet)$ is generated by its sub-FI-module $H^*(M^\bullet; \mathbb{Q}) \otimes G$. Theorem 2.74 now

implies that E_2 itself is of finite type. Since E_∞ is a subquotient of E_2 , it follows from Theorem 2.60 that E_∞ is of finite type. Since $H^i(\text{Conf}(M); \mathbb{Q})$ has a finite-length filtration whose graded pieces are of the form $E_\infty^{p, i-p}$, the theorem follows by repeatedly applying Proposition 2.17. \square

Combined with Theorem 1.14 above, Theorem 4.1 yields the theorem of Church [Ch] that the sequence of S_n -representations $\{H^i(\text{Conf}_n(M); \mathbb{Q})\}$ is uniformly representation stable. By applying this result to the trivial representation, Church extended the classical theorem on rational homological stability for the configuration spaces of *unordered* points in an open manifold to configuration spaces of unordered points in an arbitrary manifold.

The key technical innovation in [Ch] was the notion of *monotonicity*, which allows representation stability to be transmitted from the initial term of a spectral sequence to its E_∞ term. We showed in the proof of Theorem 1.14 that, for FI-modules, finite generation implies monotonicity. Thus it is not surprising that we are able to imitate the argument of [Ch] in the present paper.

4.2 Bounds on stability degree

The results of [Ch] not only show that $H^i(\text{Conf}_n(M); \mathbb{Q})$ is representation stable, but provide an explicit stable range. In this section we strengthen those results by computing explicit bound on the stability degree for the FI-module $H^i(\text{Conf}(M); \mathbb{Q})$.

Recall from Definition 2.35 that we divided the notion of stability degree into *injectivity degree* and *surjectivity degree*. For succinctness, we say that an FI-module has *stability type* (M, N) if it has injectivity degree $\leq M$ and surjectivity degree $\leq N$; note this implies it has stability degree $\leq \max(M, N)$. We recall that when V is an FI \sharp -module, whence of the form $\bigoplus M(W)$, the injectivity degree of V is 0 by Proposition 2.38.

Theorem 4.2. *Let M be a connected, oriented manifold of dimension $d \geq 3$. For any $i \geq 0$, the FI-module $H^i(\text{Conf}(M); \mathbb{Q})$ has weight $\leq i$ and stability type $(i+2-d, i)$; in particular it has stability degree $\leq i$.*

Proof. For the assertion on weight it suffices to prove that E_2^{pq} has weight $\leq p+q$, since weight is preserved under subquotients and extensions. For simplicity, let $D = d-1$. From the explicit presentation given by Totaro one sees that $E_2^{p, qD}$ is an FI \sharp -module. It is a quotient of the bidegree (p, qD) part of $H^*(M^n; \mathbb{Q})[\{G_{ij}\}]$, which has weight at most $p+2q$; so $E_2^{p, qD}$ has weight at most $p+2q$ as well. In fact, [Ch, §3.3] gives an explicit description of $E_2^{p, qD}$ as a direct sum of FI-modules of the form $M(W_i)$ where W_i is a certain representation of S_i with $i \leq p+2q$. By Proposition 2.38, this implies that $E_2^{p, qD}$ has stability type $(0, p+2q)$.

For the bound on stability type, the argument proceeds by induction on the pages of the Leray spectral sequence. To be precise, we will prove that for every $k \geq 2$ we have:

- $E_{kD+1}^{p, qD}$ has injectivity degree $\leq p+2q+1-D$;
- $E_{kD+1}^{p, qD}$ has surjectivity degree $\leq p+2q+\min(k-2, q-1)(D-2)$ for all $q > 0$;
- $E_{kD+1}^{p, 0}$ has surjectivity degree $\leq p$.

This implies that $E_\infty^{p, qD}$ has stability type $(p+2q+1-D, p+Dq+2-D)$ when $q > 0$, and that $E_\infty^{p, 0}$ has stability type $(p+2-d, p)$. This gives bounds on the graded pieces of the filtration that computes $H^i(\text{Conf}(M); \mathbb{Q})$, and the theorem then follows at once from Proposition 2.46.

The first differential with the potential not to vanish is the differential on the E_{D+1} page. In particular $E_2^{pq} = E_{D+1}^{pq}$, so $E_{D+1}^{p,qD}$ has stability type $(0, p + 2q)$. The next page of interest is $E_{2D+1} = E_{D+2}$, which is computed as the cohomology at the middle of the complex

$$E_2^{p-D-1, (q+1)D} \rightarrow E_2^{p, qD} \rightarrow E_2^{p+D+1, (q-1)D}$$

whose terms have stability type

$$(0, p + 2q + 1 - D), (0, p + 2q), \text{ and } (0, p + 2q + D - 1)$$

respectively. By Proposition 2.45, the cohomology E_{2D+1}^{pq} has stability type $(p + 2q + 1 - D, p + 2q)$. So the induction hypothesis holds for $k = 2$.

Now suppose the induction hypothesis holds for $E_{kD+1}^{p, qD}$. Again, we can write $E_{(k+1)D+1}^{p, qD}$ as the cohomology of a three-term complex:

$$E_{kD+1}^{p-kD-1, (q+k)D} \rightarrow E_{kD+1}^{p, qD} \rightarrow E_{kD+1}^{p+kD+1, (q-k)D} \quad (24)$$

By induction, the first term has surjectivity degree at most

$$p - kD - 1 + 2(q + k) + (k - 2)(D - 2) = p + 2q + 3 - 2D$$

and the second has injectivity degree $\leq p + 2q + 1 - D$. Proposition 2.45 implies that the injectivity degree of $E_{(k+1)D+1}^{p, qD}$ is at most the maximum of these quantities, which is $p + 2q + 1 - D$ since $D \geq 2$. This shows that the first item in the induction hypothesis is satisfied.

By induction, the third term has injectivity degree at most

$$p + kD + 1 + 2(q - k) + 1 - D = p + 2q + (k - 1)(D - 2)$$

when $k \leq q$, and 0 otherwise (since this term lies outside the first quadrant when $k > q$). The second term has surjectivity degree $\leq p + 2q + \min(k - 2, q - 1)(D - 2)$ when $q > 0$, and $\leq p$ when $q = 0$.

When $k \leq q$, another application of Proposition 2.45 shows that the surjectivity degree of $E_{(k+1)D+1}^{p, qD}$ is at most

$$\begin{aligned} \max(p + 2q + \min(k - 2, q - 1)(D - 2), p + 2q + (k - 1)(D - 2)) \\ = p + 2q + (k - 1)(D - 2) \\ = p + 2q + \min(k - 1, q - 1)(D - 2) \end{aligned}$$

so that the induction hypothesis is satisfied.

When $k > q > 0$, the surjectivity degree of $E_{(k+1)D+1}^{p, qD}$ is at most

$$\max(p + 2q + (q - 1)(D - 2), 0) = p + 2q + \min(k - 1, q - 1)(D - 2)$$

which again satisfies the induction hypothesis. Finally, if $q = 0$, the surjectivity degree of $E_{(k+1)D+1}^{p, qD}$ is $\leq \max(p, 0) = p$. This completes the proof. \square

Applying Theorem 4.2 together with Theorem 2.67 we obtain the following result, mentioned in the introduction.

Theorem 4.3 (Polynomiality of characters for cohomology of $\text{Conf}_n(M)$). *Let M be a connected, oriented manifold of dimension ≥ 3 with $\dim_{\mathbb{Q}}(H^*(M; \mathbb{Q})) < \infty$. Then for each $i \geq 0$ there is a character polynomial $P_{i,M}$ of degree $\leq i$ so that*

$$\chi_{H^i(\text{Conf}_n(M); \mathbb{Q})}(\sigma) = P_{i,M}(\sigma)$$

for all $n \geq 2$ and all $\sigma \in S_n$.

In particular the rational Betti number $b_i(\text{Conf}_n(M))$ agrees for all $n \geq 2i$ with a polynomial in n of degree i .

Remark 4.4. If M is a connected oriented manifold of dimension 2 with $\dim_{\mathbb{Q}}(H^*(M; \mathbb{Q})) < \infty$, the weight of the FI-module $H^i(\text{Conf } M; \mathbb{Q})$ is bounded by that of $\bigoplus_i E_2^{p, i-p}$, just as in the proof of Theorem 4.2. This weight is $p+2(i-p)$, which is maximized when $p = 0$. So in this case $H^i(\text{Conf } M; \mathbb{Q})$ is a finitely generated FI-module of weight at most $2i$, whence the characters and Betti numbers of $H^i(\text{Conf}_n(M); \mathbb{Q})$ are eventually polynomial of degree $\leq 2i$.

A “classical” application. We can apply Theorem 4.2 to prove a cohomological stability result, in the usual sense, for unordered configuration spaces with some population of colored points. To be precise, if $\mu = (\mu_1, \dots, \mu_k)$ is a partition, we denote by $B_{n,\mu}(M)$ the configuration space of sets of n distinct unordered points on M , with μ_i of the points labeled with color i and $n - |\mu|$ of the points left uncolored.

Corollary 4.5. *Let M be a connected, oriented manifold of dimension $d \geq 3$. Then*

$$H^i(B_{n,\mu}(M); \mathbb{Q}) \simeq H^i(B_{n+1,\mu}(M); \mathbb{Q})$$

for all $n > i + |\mu|$.

Corollary 4.5 is a direct improvement on [Ch, Theorem 5], where the bound $n \geq 2 \max(i, |\mu|)$ was proved. It is the bound on stability degree in Theorem 4.2 that allows us to improve the stable range from $2 \max(i, |\mu|)$ to $i + |\mu|$.

Proof. The coinvariant subspace $H^i(\text{Conf}_n(M); \mathbb{Q})_{S_{n-|\mu|}}$ carries an action of $S_{|\mu|}$, and the dimension of $H^i(B_{n,\mu}(M); \mathbb{Q})$ is the dimension of the space fixed by the action of

$$S_{\mu} := S_{\mu_1} \times \dots \times S_{\mu_k} \subset S_{|\mu|}.$$

The statement that $H^i(\text{Conf}_n(M); \mathbb{Q})$ has stability degree $\leq i$ means by definition that the isomorphism class of $H^i(\text{Conf}_n(M); \mathbb{Q})_{S_{n-|\mu|}}$ as an $S_{|\mu|}$ -representation stabilizes once $n = i + |\mu|$, whence so does the dimension of the $S_{|\mu|}$ -invariant subspace. \square

4.3 Manifolds with boundary

When M is the interior of a compact manifold with nonempty boundary, the configuration space $\text{Conf}(M)$ is in fact an $\text{FI}\sharp$ -module, at least up to homotopy.

Proposition 4.6. *Let M be the interior of a connected, oriented compact manifold \overline{M} of dimension $d \geq 2$ with nonempty boundary $\partial \overline{M}$. Then $\text{Conf}(M)$ is a homotopy $\text{FI}\sharp$ -space, i.e. a functor $\text{FI}\sharp \rightarrow \text{hTop}$.*

Proof. The fact that $\partial\overline{M} \neq \emptyset$ lets us define, for any inclusion of finite sets $B \subseteq S$, a map

$$\Psi_{B,S}: \text{Conf}_B(M) \rightarrow \text{Conf}_S(M)$$

which “adds points at infinity”, as follows. Fix a collar neighborhood R of one component of $\partial\overline{M}$, so that in particular R is connected, and fix a homeomorphism

$$\varphi: M \simeq M \setminus R.$$

Let $g: (S \setminus B) \rightarrow R$ be any configuration $g \in \text{Conf}_{S \setminus B}(R)$. Then any embedding $c: B \rightarrow M$ in $\text{Conf}_B(M)$ can be extended to a function $\Psi_{B,S}(c): S \rightarrow M$ by defining

$$\Psi_{B,S}(c)(x) = \begin{cases} \varphi(c(x)) & x \in B \\ g(x) & x \in S \setminus B \end{cases}$$

Since $\text{Conf}_{S \setminus B}(R)$ is connected, different choices of $g \in \text{Conf}_{S \setminus B}(R)$ induce homotopic maps, so the map $\Psi_{B,S}$ is well-defined up to homotopy.

Note in particular that these maps make $\text{Conf}(M)$ into a homotopy FI-space; that is, a functor from $\text{FI} \rightarrow \text{hTop}$. We have already explained above that $\text{Conf}(M)$ is a co-FI-space. Proving that $\text{Conf}(M)$ is a homotopy $\text{FI}\sharp$ -space requires a bit more. Recall that a morphism in $\text{Hom}_{\text{FI}\sharp}(S, T)$ is a triple (A, B, ϕ) with $A \subseteq S, B \subseteq T$ and $\phi: A \rightarrow B$ a bijection. To prove the proposition we must functorially associate to any such morphism a morphism

$$(A, B, \phi)_*: \text{Conf}_S(M) \rightarrow \text{Conf}_T(M)$$

We do this by defining $(A, B, \phi)_*$ to be the composition

$$\text{Conf}_S(M) \xrightarrow{|_A} \text{Conf}_A(M) \xrightarrow{\circ\phi^{-1}} \text{Conf}_B(M) \xrightarrow{\Psi_{B,T}} \text{Conf}_T(M)$$

which takes an embedding $c: S \rightarrow M$, restricts it to A , precomposes this with ϕ^{-1} , and then extends it by the map $\Psi_{B,T}$ defined above. It is straightforward to check that $(A, B, \phi) \mapsto (A, B, \phi)_*$ is functorial, thus proving the proposition. \square

Theorem 4.7. *Let M be a connected, oriented manifold which is the interior of a compact manifold with nonempty boundary. Then $H^*(\text{Conf}(M); k)$ is a graded $\text{FI}\sharp$ -algebra over k . In particular, the maps $H^*(\text{Conf}_n(M); k) \rightarrow H^*(\text{Conf}_{n+1}(M); k)$ are injective for all n . If k is a Noetherian ring of finite Krull dimension, then $H^i(\text{Conf}(M); k)$ is a finitely generated $\text{FI}\sharp$ -module.*

Proof. Proposition 4.6 implies that for any ring k , the cohomology $H^i(\text{Conf}(M); k)$ is an $\text{FI}\sharp$ -module. The injectivity of the maps $H^*(\text{Conf}_n(M); k) \rightarrow H^*(\text{Conf}_{n+1}(M); k)$ are then an immediate consequence of Proposition 2.21. We recall from the proof of Theorem 4.2 that each entry E_2^{pq} of the Leray spectral sequence $\text{Conf}_n(M) \hookrightarrow M^n$ is an $\text{FI}\sharp$ -module. Moreover, since M is homotopy equivalent to a compact manifold, its cohomology $H^i(M; k)$ is finitely generated for all $i \geq 0$, which implies that E_2^{pq} is a finitely generated $\text{FI}\sharp$ -module. However, even though this spectral sequence is known to converge to $H^*(\text{Conf}_n(M); k)$, we cannot conclude that $H^i(\text{Conf}_n(M); k)$ is finitely generated in general. The problem is that $\text{Conf}(M)$ is only a *homotopy* $\text{FI}\sharp$ -space, and the Leray spectral sequence is not homotopy invariant. As a result it is not a spectral sequence of $\text{FI}\sharp$ -modules.

However, if k is a Noetherian ring of finite Krull dimension, we can avoid this issue. We first explain the proof in the cases when k is either \mathbb{Z} or a field, which were mentioned in the introduction, and afterwards explain how to get the general case. First assume that k is a field. From the proof

of Theorem 4.2, each entry E_2^{pq} of the Leray spectral sequence is a finitely generated $\text{FI}\sharp$ -module generated in degree $\leq p + 2q$. In particular, $\dim_k E_2^{pq}(n) = O(n^{p+2q})$. Since $E_\infty^{p,q}(n)$ is a subquotient of $E_2^{pq}(n)$, we have $\dim_k E_\infty^{pq}(n) = O(n^{p+2q})$ as well. Finally, $H^i(\text{Conf}_n(M); k)$ has a filtration whose associated graded is $\bigoplus_{p+q=i} E_\infty^{pq}(n)$, so we conclude that $\dim_k H^i(\text{Conf}_n(M); k) = O(n^{2i})$. By Corollary 2.27, this implies that $H^i(\text{Conf}(M); k)$ is a finitely generated $\text{FI}\sharp$ -module, as desired.

The same argument works when $k = \mathbb{Z}$, using the fact that the rank of a \mathbb{Z} -module decreases when passing to submodules. Of course, this fact is not true for a general ring. For example, $\mathbb{C}[x, y]$ has rank 1 as a module over itself, but the ideal (x, y) is a submodule of rank 2. However, Forster [Fo, Satz 1] proved that if k is a Noetherian ring of Krull dimension d , and A is a k -module so that for every maximal ideal $\mathfrak{m} \subset k$ the k/\mathfrak{m} -vector space $A/\mathfrak{m}A$ has dimension at most N , then A is generated by at most $N + d$ elements. We want to apply this theorem to $H^i(\text{Conf}_n(M); k)$. We saw above that $E_2^{pq}(n)/\mathfrak{m}E_2^{pq}(n)$ has dimension $O(n^{p+2q})$, and since k/\mathfrak{m} is a field, this passes to subquotients and we conclude that $\dim_{k/\mathfrak{m}} H^i(\text{Conf}_n(M); k)/\mathfrak{m}H^i(\text{Conf}_n(M); k) = O(n^{2i})$ for all maximal ideals \mathfrak{m} . (Despite the use of big- O notation, we in fact have an explicit upper bound on this dimension which is independent of \mathfrak{m} , coming from $\text{rank}_{\mathbb{Z}} E_2^{pq}(n)_{/\mathbb{Z}}$.) Applying Forster's theorem, we conclude that the k -module $H^i(\text{Conf}_n(M); k)$ is generated by $O(n^{2i} + d) = O(n^{2i})$ elements. The last equivalence of Corollary 2.27 then implies that $H^i(\text{Conf}(M); k)$ is a finitely generated $\text{FI}\sharp$ -module, as desired. \square

The following theorem is now immediate from Theorem 4.2 and Remark 4.4, together with the classification of $\text{FI}\sharp$ -modules in Theorem 2.24.

Theorem 4.8 (Betti numbers are polynomial). *Let M be a connected oriented manifold with $\dim M \geq 2$ which is the interior of a compact manifold with nonempty boundary. Each of the following invariants of $\text{Conf}_n(M)$ is given by a single polynomial in n for all $n \geq 0$ (of degree $\leq i$ if $\dim M \geq 3$, and degree $\leq 2i$ if $\dim M = 2$):*

1. the i -th rational Betti number $b_i(\text{Conf}_n(M))$
2. the i -th mod- p Betti number of $\text{Conf}_n(M)$
3. the rank of $H^i(\text{Conf}_n(M); \mathbb{Z})$
4. the rank of the p -torsion part of $H^i(\text{Conf}_n(M); \mathbb{Z})$

5 Applications: cohomology of moduli spaces

5.1 Stability for $\mathcal{M}_{g,n}$ and its tautological ring

We begin by recalling some basic facts about moduli spaces of Riemann surfaces and their cohomology.

Let S_g be a closed, oriented surface of genus $g \geq 2$. For each $n \geq 1$ let (y_1, \dots, y_n) be an ordered n -tuple of n distinct points on S_g . Let $\mathcal{M}_{g,n}$ denote the moduli space of n -pointed Riemann surfaces $(X; y_1, \dots, y_n)$ homeomorphic to $(S_g; y_1, \dots, y_n)$. The space $\mathcal{M}_{g,n}$ has the structure of a complex orbifold, i.e. the quotient of a complex manifold by a finite group of holomorphic automorphisms.

The map “forget the n^{th} point” yields a fibration $p_n: \mathcal{M}_{g,n} \rightarrow \mathcal{M}_{g,n-1}$ whose fiber is an $(n-1)$ -pointed surface of genus g . For each $k \geq 0$, the map p_n induces a homomorphism

$$p_n^*: H^k(\mathcal{M}_{g,n-1}; \mathbb{Q}) \rightarrow H^k(\mathcal{M}_{g,n}; \mathbb{Q}).$$

For each $i = 1, \dots, n$, we let $L_i \rightarrow \mathcal{M}_{g,n}$ be the complex line bundle over $\mathcal{M}_{g,n}$ whose fiber at $(X; y_1, \dots, y_n)$ is the cotangent space to X at y_i . Let $\psi_i = c_1(L_i) \in H^2(\mathcal{M}_{g,n}; \mathbb{Q})$ be the first Chern

class of the line bundle L_i . Integration along the fiber yields a Gysin homomorphism

$$(p_n)! : H^j(\mathcal{M}_{g,n}; \mathbb{Q}) \rightarrow H^{j-2}(\mathcal{M}_{g,n-1}; \mathbb{Q}).$$

For each $j \geq 0$ define $\kappa_j := (p_{n+1})!(\psi_{n+1}^{j+1}) \in H^{2j}(\mathcal{M}_{g,n}; \mathbb{Q})$.

The *tautological ring* $\mathcal{R}(\mathcal{M}_{g,n})$ is defined to be the subring of $H^*(\mathcal{M}_{g,n}; \mathbb{Q})$ generated by

$$\{\psi_i \mid i \geq 0\} \cup \{\kappa_j \mid j \geq 0\}.$$

This ring has been intensively studied by algebraic geometers (see, e.g., [Va]). The usual grading on $\mathcal{R}(\mathcal{M}_{g,n})$ is half the cohomological grading, so that ψ_i has grading 1 and κ_j has grading j .

S_n acts on $\mathcal{M}_{g,n}$ by permuting the marked points. The induced action on $H^*(\mathcal{M}_{g,n}; \mathbb{Q})$ satisfies

$$\sigma \cdot \kappa_j = \kappa_j \quad \text{and} \quad \sigma \cdot \psi_i = \psi_{\sigma(i)}$$

for each $j \geq 1$ and each $1 \leq i \leq n$, so $\mathcal{R}(\mathcal{M}_{g,n})$ is a subrepresentation of $H^*(\mathcal{M}_{g,n}; \mathbb{Q})$. The degree j components $\mathcal{R}^j(\mathcal{M}_{g,n})$ can be quite complicated as S_n -representations, and for $g \geq 3$ they are poorly understood. However, we have the following strong constraint once n is sufficiently large.

Theorem 5.1. *For each $g \geq 2$ the tautological ring $\mathcal{R}(\mathcal{M}_{g,\bullet})$ is a graded FI-algebra of finite type. Thus for each $j \geq 1$ the characters $\chi_{\mathcal{R}^j(\mathcal{M}_{g,n})}$ are eventually polynomial of degree $\leq j$. In particular, $\dim \mathcal{R}^j(\mathcal{M}_{g,n})$ is eventually polynomial in n of degree $\leq j$.*

Proof. The collection of spaces $\mathcal{M}_{g,\bullet}$ form a co-FI-space just as in §4.1. Thus the cohomology $H^*(\mathcal{M}_{g,\bullet}; \mathbb{Q})$ is a graded FI-algebra. Let V be the graded sub-FI-module spanned by the ψ_i and κ_j . The only nonzero graded pieces of V are

$$V^2 \simeq M(1) \oplus M(0), \quad V^{2i} = \mathbb{Q}\kappa_i \simeq M(0) \text{ for } i > 1,$$

so V is a graded FI-module of finite type and has slope $\leq \frac{1}{2}$. Theorem 2.74 implies that the sub-FI-algebra of $H^*(\mathcal{M}_{g,\bullet}; \mathbb{Q})$ generated by V is a graded FI-module of finite type which has slope $\leq \frac{1}{2}$. But this subalgebra is, by definition, nothing other than $\mathcal{R}(\mathcal{M}_{g,n})$. We conclude that the dual of $\mathcal{R}^j(\mathcal{M}_{g,\bullet}) \subset H^{2j}(\mathcal{M}_{g,\bullet}; \mathbb{Q})$ is an FI-module, finitely generated in degree $\leq i$, so the desired conclusion follows from Theorem 2.67. \square

Jimenez Rolland [J1] proved the related theorem that for each fixed $g \geq 2$ and $i \geq 0$, the sequence $\{H^i(\mathcal{M}_{g,n}; \mathbb{Q})\}$ is a uniformly representation stable sequence of S_n -representations. Since $H^i(\mathcal{M}_{g,\bullet}; \mathbb{Q})$ is an FI-module, and since $H^*(\mathcal{M}_{g,n}; \mathbb{Q})$ is finite dimensional, Theorem 1.14 together with Jimenez Rolland's theorem shows that $\{H^*(\mathcal{M}_{g,\bullet}; \mathbb{Q})\}$ is a graded FI-module of finite type. This result, with bounds on the stability degree, etc., has been worked out by Jimenez Rolland [J2] using the theorems in the present paper.

5.2 Albanese cohomology of Torelli groups and variants

The Torelli subgroups \mathcal{I}_g^1 and IA_n of the mapping class groups and automorphism groups of free groups, respectively, are of great interest in low-dimensional topology (see below for definitions). However, almost nothing is known about $H^*(\mathcal{I}_g^1; \mathbb{Q})$ or $H^*(\text{IA}_n; \mathbb{Q})$. In this section we apply the theory of FI-modules to the subalgebra generated by cohomology in degree 1.

Definition 5.2 (Albanese cohomology). Let Γ be a finitely generated group, let Γ^{ab} be its abelianization, and let $\psi: \Gamma \rightarrow \Gamma^{\text{ab}}$ be the natural quotient map. Since $H^1(\Gamma^{\text{ab}}; \mathbb{Q}) \simeq H^1(\Gamma; \mathbb{Q})$ and $H^*(\Gamma^{\text{ab}}; \mathbb{Q}) \simeq \bigwedge^* H^1(\Gamma^{\text{ab}}; \mathbb{Q})$, the map ψ induces a homomorphism

$$\psi^*: H^*(\Gamma^{\text{ab}}; \mathbb{Q}) = \bigwedge^* H^1(\Gamma; \mathbb{Q}) \rightarrow H^*(\Gamma; \mathbb{Q}).$$

We define the *Albanese cohomology* $H_{\text{Alb}}^*(\Gamma; \mathbb{Q})$ of Γ to be the image of this map:

$$H_{\text{Alb}}^*(\Gamma; \mathbb{Q}) := \psi^*(H^*(\Gamma^{\text{ab}}; \mathbb{Q})) \subset H^*(\Gamma; \mathbb{Q})$$

We use the name “Albanese cohomology” in order to keep in mind the frequently encountered case where Γ is the fundamental group of a compact Kahler manifold, in which case H_{Alb}^* is the part of the cohomology coming from the associated Albanese variety. Clearly $H_{\text{Alb}}^*(\Gamma; \mathbb{Q})$ can also be described as the subalgebra of $H^*(\Gamma; \mathbb{Q})$ generated by $H^1(\Gamma; \mathbb{Q})$.

Albanese cohomology of the Torelli group. Let S_g^1 be a compact, oriented genus $g \geq 2$ surface with one boundary component. The mapping class group $\text{Mod}(S_g^1)$ is the group of path components of the group $\text{Homeo}^+(S_g^1, \partial S_g^1)$ of orientation-preserving homeomorphisms of S_g^1 fixing the boundary pointwise. The action of $\text{Mod}(S_g^1)$ on $H_1(S_g^1; \mathbb{Z})$ preserves algebraic intersection number, which is a symplectic form on $H_1(S_g^1; \mathbb{Z})$. The *Torelli group* \mathcal{I}_g^1 is defined to be the kernel of this action, and we have a well-known (see, e.g. [FM]) exact sequence, where $\text{Sp}_{2g} \mathbb{Z}$ is the integral symplectic group:

$$1 \rightarrow \mathcal{I}_g^1 \rightarrow \text{Mod}(S_g^1) \rightarrow \text{Sp}_{2g} \mathbb{Z} \rightarrow 1$$

Very little is known about the cohomology $H^*(\mathcal{I}_g^1; \mathbb{Q})$ of the Torelli group, or even about the subalgebra $H_{\text{Alb}}^*(\mathcal{I}_g^1; \mathbb{Q})$. Johnson proved that

$$H_{\text{Alb}}^1(\mathcal{I}_g^1; \mathbb{Q}) = H^1(\mathcal{I}_g^1; \mathbb{Q}) \simeq \bigwedge^3 \mathbb{Q}^{2g}$$

as $\text{Sp}_{2g} \mathbb{Z}$ -modules. Hain [Ha] computed $H_{\text{Alb}}^2(\mathcal{I}_g^1; \mathbb{Q})$ as an $\text{Sp}_{2g} \mathbb{Z}$ -module, and Sakasai [Sa] did the same for $H_{\text{Alb}}^3(\mathcal{I}_g^1; \mathbb{Q})$. Many examples of nontrivial classes in $H_{\text{Alb}}^*(\mathcal{I}_g^1; \mathbb{Q})$ were found in [CF2]; in particular a lower bound of $O(g^{i+2})$ for the dimension of $H_{\text{Alb}}^i(\mathcal{I}_g^1; \mathbb{Q})$ was proved. Beyond these coarse bounds, nothing is known about the precise dimension of $H_{\text{Alb}}^i(\mathcal{I}_g^1; \mathbb{Q})$.

Theorem 5.3. *For each $i \geq 0$ the dimension of $H_{\text{Alb}}^i(\mathcal{I}_g^1; \mathbb{Q})$ is polynomial in g , of degree at most $3i$, for $g \gg i$.*

Although Johnson’s theorem implies that $\dim H_{\text{Alb}}^i(\mathcal{I}_g^1; \mathbb{Q})$ grows no faster than $O(g^{3i})$, the statement that the dimension of $H_{\text{Alb}}^i(\mathcal{I}_g^1; \mathbb{Q})$ coincides exactly with a polynomial for large g is much stronger, requiring some understanding of the relations between wedge products of the classes in H^1 . We emphasize that the dimension of $H_{\text{Alb}}^i(\mathcal{I}_g^1; \mathbb{Q})$ is unknown when $i \geq 3$; in particular we do not know what the polynomials produced by Theorem 5.3 are.

In order to prove Theorem 5.3 we will prove that $H_{\text{Alb}}^*(\mathcal{I}_{\bullet}^1; \mathbb{Q})$ is a graded FI-module of finite type. One novelty here is that the FI-module structure on $H_{\text{Alb}}^*(\mathcal{I}_{\bullet}^1; \mathbb{Q})$ is in some sense not natural, but its existence nevertheless allows us to prove Theorem 5.3.

Proof of Theorem 5.3. Fix for each g a symplectic basis $\{a_1, b_1, \dots, a_g, b_g\}$ for $H_1(S_g^1; \mathbb{Z})$. Any injection $f: \{1, \dots, m\} \hookrightarrow \{1, \dots, n\}$ determines an inclusion $f_*: H_1(S_m^1; \mathbb{Z}) \hookrightarrow H_1(S_n^1; \mathbb{Z})$ by $f_*(a_i) = a_{f(i)}$ and $f_*(b_i) = b_{f(i)}$. Any inclusion $S_m^1 \hookrightarrow S_n^1$ determines an inclusion $H_1(S_m^1; \mathbb{Z}) \hookrightarrow H_1(S_n^1; \mathbb{Z})$. Such an inclusion also determines an inclusion $\mathcal{I}_m^1 \hookrightarrow \mathcal{I}_n^1$, given by extending each map to be the identity in the complement. Moreover, although there are many inclusions $S_m^1 \hookrightarrow S_n^1$ inducing a given map on homology, Johnson [Jo2, Theorem 1A] proved that the resulting inclusions $\mathcal{I}_m^1 \hookrightarrow \mathcal{I}_n^1$ are all conjugate in

\mathcal{I}_n^1 . In particular, they all induce the same map $H_*(\mathcal{I}_m^1) \rightarrow H_*(\mathcal{I}_n^1)$. Thus we can define the structure of a graded FI-module on $H_*(\mathcal{I}_\bullet^1; \mathbb{Q})$ by letting $f_*: H_*(\mathcal{I}_m^1; \mathbb{Q}) \rightarrow H_*(\mathcal{I}_n^1; \mathbb{Q})$ be the map induced by some (equivalently, any) inclusion $S_m^1 \hookrightarrow S_n^1$ which on homology sends $a_i \mapsto a_{f(i)}$ and $b_i \mapsto b_{f(i)}$. Of course, this FI-module structure is not unique; it depends on the initial choice of basis for $H_1(S_n^1; \mathbb{Z})$. But all that matters for us is the existence of an FI-module structure on $H_*(\mathcal{I}_\bullet^1; \mathbb{Q})$ with respect to which $H_1(\mathcal{I}_\bullet^1; \mathbb{Q})$ is finitely generated.

As mentioned above, $H^1(\mathcal{I}_g^1; \mathbb{Q})$ is isomorphic to $\bigwedge^3 \mathbb{Q}^{2g}$ as a representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$. Restricting to the standard copy of the subgroup S_g of $\mathrm{Sp}_{2g} \mathbb{Z}$ that sits inside the Weyl group, we find that $H_1(\mathcal{I}_\bullet^1; \mathbb{Q})$ is isomorphic as an FI-module to $\bigwedge^3(M(1) \oplus M(1))$. In particular, $H_1(\mathcal{I}_\bullet^1; \mathbb{Q})$ is a finitely generated FI-module of weight 3 (e.g. by Proposition 2.62.) Proposition 2.77 now shows that $H_{\mathrm{Alb}}^*(\mathcal{I}_\bullet^1; \mathbb{Q})$ is a co-FI-algebra of finite type, so the desired conclusion follows from Theorem 2.67. \square

Albanese cohomology of IA_n . Let F_n denote the free group of rank n . The Torelli subgroup IA_n of $\mathrm{Aut}(F_n)$ is defined to be the subgroup consisting of those automorphisms that act trivially on $H_1(F_n; \mathbb{Z}) \simeq \mathbb{Z}^n$, giving the exact sequence

$$1 \rightarrow \mathrm{IA}_n \rightarrow \mathrm{Aut}(F_n) \rightarrow \mathrm{GL}_n \mathbb{Z} \rightarrow 1.$$

Magnus proved in 1934 that IA_n is finitely generated. The conjugation action of $\mathrm{Aut}(F_n)$ on IA_n induces an action of $\mathrm{GL}_n \mathbb{Z}$ on the homology groups $H_i(\mathrm{IA}_n; \mathbb{Q})$. Farb, Cohen–Pakianathan and Kawazumi (see e.g. [Ka]) independently proved that

$$H_1(\mathrm{IA}_n; \mathbb{Q}) \simeq \bigwedge^2 \mathbb{Q}^n \otimes (\mathbb{Q}^n)^* \quad (25)$$

as $\mathrm{GL}_n \mathbb{Z}$ -representations, where the action on \mathbb{Q}^n is the restriction of the standard $\mathrm{GL}_n \mathbb{Q}$ -representation. The isomorphism is provided by the Johnson homomorphism, which is compatible not only with the S_n -actions on both sides, but with the natural FI-structures that both sides acquire as n varies. As with the Torelli group, basically nothing is known about $H_{\mathrm{Alb}}^*(\mathrm{IA}_n; \mathbb{Q})$; other than $H_{\mathrm{Alb}}^1 = H^1$, the only progress is Pettet’s computation [Pe] of $H_{\mathrm{Alb}}^2(\mathrm{IA}_n; \mathbb{Q})$.

Theorem 5.4. *For each $i \geq 0$ the dimension of $H_{\mathrm{Alb}}^i(\mathrm{IA}_n; \mathbb{Q})$ is polynomial in n , of degree at most $3i$, for $n \gg i$.*

Proof. The proof proceeds along much the same lines as that of Theorem 5.3. For any injection $f: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ we can define an injection

$$f_*: \mathrm{IA}_m \hookrightarrow \mathrm{IA}_n$$

as follows. Let $f_*: F_m \hookrightarrow F_n$ denote the inclusion induced by $x_i \mapsto x_{f(i)}$. Given $\varphi \in \mathrm{IA}_m$, we define $f_*\varphi \in \mathrm{IA}_n$ to be the automorphism sending $x_{f(i)} \mapsto f_*(\varphi(x_i))$ and $x_j \mapsto x_j$ for those j not lying in the image of f . It is easy to check that $(f \circ g)_* = f_* \circ g_*$, so the induced maps on homology make $H_*(\mathrm{IA}_\bullet; \mathbb{Q})$ into a graded FI-module.

Considering (25) as a representation of S_n , we compute that the FI-module $H_1(\mathrm{IA}_\bullet; \mathbb{Q})$ can be written as

$$\bigwedge^2 M(1) \otimes M(1) \simeq M(\square\square) \oplus M(\begin{smallmatrix} \square & \\ & \square \end{smallmatrix}) \oplus M(\begin{smallmatrix} \square & \square \\ & \square \end{smallmatrix}) \oplus M(\begin{smallmatrix} \square & & \\ & \square & \\ & & \square \end{smallmatrix}).$$

In particular, we see that $H_1(\mathrm{IA}_\bullet; \mathbb{Q})$ is finitely generated as an FI-module and has weight 3 (we could also deduce this from Propositions 2.61 and 2.62). Thus Corollary 2.80 implies that the dual of the co-FI-module $H_{\mathrm{Alb}}^*(\mathrm{IA}_\bullet; \mathbb{Q})$ is a graded FI-module of finite type. Theorem 2.67 now implies the theorem, and moreover that the character of $H_{\mathrm{Alb}}^i(\mathrm{IA}_n; \mathbb{Q})$ as an S_n -representation is eventually given by a character polynomial. \square

5.3 Graded Lie algebras associated to the lower central series

Recall that the *lower central series*

$$\Gamma = \Gamma_1 > \Gamma_2 > \cdots$$

of a group Γ is defined inductively by $\Gamma_1 := \Gamma$ and $\Gamma_{j+1} := [\Gamma, \Gamma_j]$. For simplicity, assume that k is a field. The *associated graded Lie algebra* of Γ , denoted $\text{gr}(\Gamma)$, is the Lie algebra over k defined by

$$\text{gr}(\Gamma) := \bigoplus_{j=1}^{\infty} \text{gr}(\Gamma)_j = \bigoplus_{j=1}^{\infty} (\Gamma_j / \Gamma_{j+1}) \otimes k$$

where the Lie bracket is induced by the group commutator.

When Γ is finitely generated the graded vector space $\text{gr}(\Gamma)$ is finite dimensional in each grading. The natural action of the automorphism group $\text{Aut}(\Gamma)$ on Γ preserves each Γ_j , and so acts on $\text{gr}(\Gamma)$. It is easy to see that this action factors through an action of $\text{Aut}(\Gamma/[\Gamma, \Gamma])$; in particular there is a well-defined action of the group $\text{Out}(\Gamma)$ of outer automorphisms of Γ on $\text{gr}(\Gamma)$. The action on $\text{gr}(\Gamma)_1$ in grading 1 is nothing more than the representation of $\text{Aut}(\Gamma/[\Gamma, \Gamma]) = \text{Aut}(H_1(\Gamma; \mathbb{Z}))$ on $H_1(\Gamma; k) \simeq H_1(\Gamma; \mathbb{Z}) \otimes k$.

Sending $X \mapsto \text{gr}(\pi_1(X))$ defines a functor from Top to the category of graded Lie algebras. This is well-defined since up to conjugation $\pi_1(X)$ does not depend on a choice of basepoint for X , and conjugation acts trivially on $\text{gr}(\pi_1(X))$. Thus for any FI-space X (or even just a homotopy FI-space), the associated graded Lie algebra $\text{gr}(\pi_1(X))$ is a graded FI-algebra. More generally, we say that Γ is an *FI-group (up to conjugacy)* if for each $f \in \text{Hom}_{\text{FI}}(\mathbf{n}, \mathbf{m})$ we have maps $f_*: \Gamma_n \rightarrow \Gamma_m$ so that the relevant diagrams commute up to conjugacy. Similarly we have the notion of *FI \sharp -group (up to conjugacy)*. If Γ is an FI-group up to conjugacy, then $\text{gr}(\Gamma)$ is a graded FI-module.

Theorem 5.5 (Finite generation of $\text{gr}(\Gamma)$). *If Γ is an FI-group up to conjugacy (e.g. the fundamental group of a homotopy FI-space) and the FI-module $H_1(\Gamma; k)$ is finitely generated, then the graded FI-module $\text{gr}(\Gamma)$ is of finite type.*

Proof. For any group Γ , the terms Γ_i of the lower central series are by definition generated by iterated commutators of elements of Γ . This shows that $\text{gr}(\Gamma)$ is always generated as a Lie algebra by $\text{gr}(\Gamma)_1 \simeq H_1(\Gamma; k)$. Thus if $H_1(\Gamma; k)$ is finitely generated, Theorem 2.74 implies that $\text{gr}(\Gamma)$ is an FI-algebra of finite type. \square

Example 5.6 (Free groups). Let Γ be the FI \sharp -group where $\Gamma_n = F_n$ is the free group of rank n , and where a morphism $\phi: A \rightarrow B$ induces the map $F_n \rightarrow F_m$ given by $\phi(x_i) = x_{\phi(i)}$ for $i \in A$, and $\phi(x_i) = 1$ for $i \notin A$. Since $\text{gr}(\Gamma)_1 \simeq M(1)$ is finitely generated, Theorem 5.5 implies that $\text{gr}(\Gamma)$ is of finite type. It is known that $\text{gr}(F_n)$ is isomorphic to the free Lie algebra on n variables. Applying Theorem 1.14, we conclude that the graded pieces of the free Lie algebra on n variables are representation stable as representations of S_n . A variant of this result, for $\text{GL}_n \mathbb{C}$ -representations rather than S_n -representations, was proved in [CF, Corollary 5.7].

Example 5.7 (Pure braid groups). We proved in Proposition 4.6 that the configuration space $\text{Conf}_n(\mathbb{R}^2)$ of ordered n -tuples of distinct points in the plane is a homotopy FI \sharp -space. Since the pure braid group P_n on n strands is the fundamental group $P_n = \pi_1(\text{Conf}_n(\mathbb{R}^2))$, this shows that P_n is an FI \sharp -group up to conjugation. We proved in Example 3.3 that $H_1(P_n; \mathbb{Q}) \simeq M(\square\square)$, so Theorem 5.5 implies that $\text{gr}(P_n)$ is a graded FI \sharp -module of finite type.

Hain [Ha] proved that there is an isomorphism of S_n -representations $\text{gr}(P_n) \simeq \text{gr}(\mathfrak{p}_n)$, where \mathfrak{p}_n is the *Malcev Lie algebra* of P_n . Applying Theorem 1.14 to Example 5.7 implies the following theorem, which confirms Conjecture 5.15 of [CF].

Theorem 5.8 (Representation stability for \mathfrak{p}_n). *For each fixed $i \geq 1$, the sequence $\{\mathfrak{p}_n^i\}$ of grading- i pieces of \mathfrak{p}_n is a uniformly representation stable sequence of S_n -representations.*

Drinfeld–Kohno (see [Ko]) actually found an explicit presentation of \mathfrak{p}_n ; for another approach to Theorem 5.8, we could apply Theorem 2.74 directly to their presentation.

Remark 5.9. The pure braid groups P_n are examples of *pseudo-nilpotent* groups, so that $H^*(\mathfrak{p}_n; \mathbb{Q}) \simeq H^*(P_n; \mathbb{Q})$ for all n . It was already proved in [CF] that the sequence $\{H^i(P_n; \mathbb{Q})\}$ is uniformly representation stable for each $i \geq 0$, so one is tempted to derive Theorem 5.8 directly from [CF, Theorem 5.3], which states the equivalence of uniform representation stability for a Lie algebra and for its (co)homology. However, that theorem was only proved in [CF] in the context of stability for $\mathrm{SL}_n \mathbb{C}$ -representations and $\mathrm{GL}_n \mathbb{C}$ -representations. Indeed the “strong stability” hypothesis assumed in that theorem almost never holds for S_n -representations (and it does not hold here).

Example 5.10 (The Torelli group). Recall from Section 5.2 the definition of the Torelli group \mathcal{I}_g^1 of a compact surface of genus $g \geq 3$ with one boundary component. The conjugation action of the mapping class group $\mathrm{Mod}(S_g^1)$ on \mathcal{I}_g^1 induces a well-defined action of $\mathrm{Sp}_{2g} \mathbb{Z}$ on $\mathrm{gr}(\mathcal{I}_g^1)$. A finite presentation for $\mathrm{gr}(\mathcal{I}_g^1)$ as a Lie algebra has been given by Habegger–Sorger [HS], extending the fundamental computation of Hain [Ha] in the closed case. Hain also worked out the first few graded terms of this Lie algebra explicitly as $\mathrm{Sp}_{2g} \mathbb{Z}$ -representations. Getzler–Garoufalidis (personal communication) have recently given more detailed computations in this direction. However, exact computations in arbitrary degrees seem out of reach. Even so, we have the following.

Theorem 5.11. *For each $i \geq 0$ the dimension $\dim(\mathrm{gr}(\mathcal{I}_g^1)_i)$ is polynomial in g for $g \gg i$.*

Proof. The discussion in Section 5.2 shows that \mathcal{I}_\bullet^1 is an FI-group up to conjugacy. We explained in Section 5.2 that $H_1(\mathcal{I}_\bullet^1; \mathbb{Q})$ is a finitely generated FI-module. Theorem 5.5 now implies that $\mathrm{gr}(\mathcal{I}_g^1)$ is a graded FI-module of finite type, and the claim follows from Theorem 2.67. \square

Theorem 5.12. *For each $i \geq 0$ the dimension $\dim(\mathrm{gr}(\mathrm{IA}_n)_i)$ is polynomial in n for $n \gg i$.*

Proof. We proved in Section 5.2 that IA_\bullet is an FI-group up to conjugacy (in fact, it is even a functor from FI to groups.) We also proved that $H_1(\mathrm{IA}_\bullet; \mathbb{Q})$ is a finitely generated FI-module, so the claim follows from Theorems 5.5 and 2.67 as above. \square

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